Quasi-locally Finite Polynomial Endomorphisms.

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Abstract.

If F is a polynomial endomorphism of $\mathbb{C}^N$, let $\mathbb{C}(X)^F$ denote the field of rational functions $r \in \mathbb{C}(x_1, \ldots, x_N)$ such that $r \circ F = r$. We will say that F is quasi-locally finite if there exists a nonzero $p \in \mathbb{C}(X)^F[T]$ such that $p(F) = 0$. This terminology comes out from the fact that this definition is less restrictive than the one of locally finite endomorphisms made in [7]. Indeed, F is called locally finite if there exists a nonzero $p \in \mathbb{C}[T]$ such that $p(F) = 0$. In the present paper, we show that F is quasi-locally finite if and only if for each $a \in \mathbb{C}^N$ the sequence $n \mapsto F^n(a)$ is a linear recurrent sequence. Therefore, this notion is in some sense natural. We also give a few basic results on such endomorphisms. For example: they satisfy the Jacobian conjecture.

Keywords.

Polynomial automorphisms, linear recurrent sequences.

INTRODUCTION.

Let us denote by $\mathbb{A}^N = \mathbb{C}^N$ the complex affine space of dimension N and by End the set of polynomial endomorphisms of $\mathbb{A}^N$. As usual, we identify an element F of End to the N-uple of its coordinate functions $F = (F_1, \ldots, F_N)$ where each F_L belongs to the ring $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_N]$ of regular functions on $\mathbb{A}^N$. We will therefore write $End = \mathbb{C}[X]^N$. Let us set $\mathbb{C}(X) := \mathbb{C}(x_1, \ldots, x_N)$, $\mathbb{C}(X)^F := \{r \in \mathbb{C}(X),\, r \circ F = r\}$ and $\mathbb{C}[X]^F := \mathbb{C}(X)^F \cap \mathbb{C}[X]$. We recall that F is called dynamically trivial if its dynamical degree $dd(F) := \lim_{n \to \infty} (\deg\, F^n)^{\frac{1}{n}}$ is equal to one (see [5]). In the case where F is an automorphism, this is equivalent to saying that its topological entropy $h(F)$ is zero (see [13]). A first subclass of dynamically trivial polynomial endomorphisms was introduced in [7]. It is the set of polynomial endomorphisms F which are locally finite (LF for short) in the following sense: the complex vector space generated by the $r \circ F^n$, $n \geq 0$, is finite dimensional for each $r \in \mathbb{C}[X]$. In the last quoted paper, it is shown that this is equivalent to saying that the sequence $n \mapsto \deg F^n$ is upper bounded or to saying that there exists a nonzero $p \in \mathbb{C}[T]$ such that $p(F) = 0$. Using a deep result from number theory known as the theorem of Skolem-Mahler-Lech (see [9, 12]), one can show that this amounts to saying that the sequence $n \mapsto \deg F^n$ is periodic for large n (in [6], the proof is given for $N = 2$, but it is easy to give a general proof).

Here, we are interested by the wider class of polynomial endomorphisms F which are quasi-locally finite (QLF for short) in the following sense: there exists a nonzero $p \in \mathbb{C}(X)^F[T]$ such that $p(F) = 0$.

Section I is devoted to generalites. We introduce the minimal polynomial $\nu_F \in \mathbb{C}(X)^F[T]$ of a QLF polynomial endomorphism F and show in prop. 1.3 that in fact $\nu_F \in \mathbb{C}[X]^F[T]$. In prop. 1.5 we show that for any QLF polynomial endomorphism F the sequence $n \mapsto \deg F^n$ has at most linear growth. Therefore, as announced, any QLF polynomial endomorphism is dynamically trivial. In section II, we prove our main theorem asserting that F is QLF if and only if the sequence $n \mapsto F^n(a)$ is a linear recurrent sequence for any $a \in \mathbb{A}^N$. In section III, we give two criteria for invertibility of QLF polynomial endomorphisms.

I. GENERALITIES.

Let $F \in End$. In [7], we noticed that $\mathcal{I}_F := \{p \in \mathbb{C}[T],\, p(F) = 0\}$ is an ideal of $\mathbb{C}[T]$. Indeed, it is a complex vector subspace of $\mathbb{C}[T]$ which is stable by multiplication by T. In the case where F is LF, i.e. when $\mathcal{I}_F \neq \{0\}$, we denote by μ_F the (unique) monic polynomial generating this ideal. By the same way, $\mathcal{I}'_F := \{p \in \mathbb{C}(X)^F[T],\, p(F) = 0\}$ is an ideal of $\mathbb{C}(X)^F[T]$. In the case where F is QLF, i.e. when $\mathcal{I}'_F \neq \{0\}$, we denote by ν_F the (unique) monic polynomial generating this ideal.

Proposition 1.1. If $F \in End$ is QLF, the following assertions are equivalent:

(i) F is LF; (ii) $\nu_F \in \mathbb{C}[T]$.

Furthermore, if these assertions are satisfied, we have $\mu_F = \nu_F$.

Proof. If F is LF, it is clear that ν_F divides μ_F in $\mathbb{C}(X)^F[T]$. Since $\mu_F \in \mathbb{C}[T]$, we clearly have $\nu_F \in \mathbb{C}[T]$. Conversely, if $\nu_F \in \mathbb{C}[T]$, then F is obviously LF. □

We introduce the language of linear recurrent sequences (LRS for short) and we refer to [3] for a nice overview of this subject. Let K be any field and let V be any vector space over K. The set of sequences $u : \mathbb{N} \to V$ will be denoted by $V^{\mathbb{N}}$. If $p = \sum_k p_k T^k \in K[T]$, we define $p(u) \in V^{\mathbb{N}}$ by the formula $\forall\, n \in \mathbb{N},\, \big(p(u)\big)(n) = \sum_k p_k\, u(n+k)$ and we set $\mathcal{I}_u := \{p \in K[T],\, p(u) = 0\}$. It is easy to show that $\mathcal{I}_u$ is an ideal of $K[T]$. We say that $u \in V^{\mathbb{N}}$ is a LRS if $\mathcal{I}_u \neq \{0\}$. In this case, the minimal polynomial of u is defined as the (unique) monic polynomial μ_u generating the ideal $\mathcal{I}_u$. If a LRS of (the vector space) K takes values in a subfield K', it is well known that its minimal polynomial belongs to $K'[T]$. More generally, we have the following result.

Lemma. If u is a LRS of a field K taking values in a subring A which is noetherian and factorial, then $\mu_u \in A[T]$.

Proof. We may assume that K is the field of fractions of A. Since A is factorial, it is sufficient to prove that $\mathcal{I}_u = \{p \in K[T],\, p(u) = 0\}$ contains a monic polynomial in $A[T]$. If $v = (v_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$, let us denote by $E(v)$ the sequence $(v_{n+1})_{n\in\mathbb{N}}$. Let M be the A-module generated by the $E^k(u)$, $k \in \mathbb{N}$. If p is a nonzero element of $\mathcal{I}_u$, it is clear that $\forall\, v \in M$, $p(v) = 0$. Therefore, if $d := \deg p$, the map $M \to A^d$, $v \mapsto (v_k)_{0\le k\le d-1}$ is injective. Since A is noetherian, this shows that M is a finite A-module. Let $m \geq 0$ be such that the $E^k(u)$, $0 \le k \le m$, generate M. There exist $\lambda_k \in A$, $0 \le k \le m$, such that $E^{m+1}(u) = \sum_{0\le k\le m} \lambda_k E^k(u)$. In other words, $T^{m+1} - \sum_{0\le k\le m} \lambda_k T^k \in \mathcal{I}_u$. □

Example. Any LRS with values in $\mathbb{Z}$ admits a minimal polynomial in $\mathbb{Z}[T]$.

The next trivial result relates QLF polynomial endomorphisms and LRS.

Proposition 1.2. If $F \in End$, the following assertions are equivalent:

(i) F is QLF;

(ii) the sequence $n \mapsto F^n$ is a LRS of $\mathbb{C}(X)^N$ considered as a vector space over $\mathbb{C}(X)^F$.

Furthermore, if these assertions are satisfied, the associated minimal polynomials are equal.

Proof. If $p = \sum_k p_k T^k \in \mathbb{C}(X)^F[T]$, $\quad \sum_k p_k\, F^k = 0 \iff \forall\, n \in \mathbb{N}, \sum_k p_k\, F^{k+n} = 0$. □

Remark. If $F \in End$, it is clear that the set of polynomials $p \in \mathbb{C}(X)[T]$ satisfying $p(F) = 0$ is a nonzero ideal of $\mathbb{C}(X)[T]$. However, it seems that there is in general no connection with LRS. Indeed, if $p = \sum_k p_k T^k \in \mathbb{C}(X)[T]$ satisfies $\sum_k p_k\, F^k = 0$, it is not necessarily true that $\forall\, n \in \mathbb{N}, \sum_k p_k\, F^{k+n} = 0$.

Proposition 1.3. If $F \in End$ is QLF, then $\nu_F \in \mathbb{C}[X]^F[T]$.

Proof. It follows from prop. 1.2 that the sequence $n \mapsto F^n$ is a LRS of the vector space $\mathbb{C}(X)^N$ over $\mathbb{C}(X)$. If $1 \le L \le N$, let us denote by $\Pi_L : \mathbb{C}(X)^N \to \mathbb{C}(X)$ the L-th projection. Each sequence $n \mapsto \Pi_L(F^n)$ being a LRS of the field $\mathbb{C}(X)$ with values in $\mathbb{C}[X]$, its minimal polynomial $\mu_{L,F}$ has coefficients in $\mathbb{C}[X]$. Since $\nu_F = \underset{1\le L\le N}{lcm}\, \mu_{L,F}$, we are done. □

Proposition 1.4. If $F \in End$, the following assertions are equivalent:

(i) F is QLF;

(ii) the sequence $n \mapsto F^n$ is a LRS of $\mathbb{C}(X)^N$ considered as a vector space over $\mathbb{C}(X)$. Furthermore, if these assertions are satisfied, the associated minimal polynomials are equal.

Proof. (i) $\Longrightarrow$ (ii) is a direct consequence of prop. 1.2. Let us show (ii) $\Longrightarrow$ (i). Let $p \in \mathbb{C}(X)[T]$ be the minimal polynomial of the sequence $n \mapsto F^n$ considered as a LRS of the vector space $\mathbb{C}(X)^N$ over $\mathbb{C}(X)$. The proof of prop. 1.3 shows that $p \in \mathbb{C}[X][T]$. It is sufficient to show that $p \in \mathbb{C}[X]^F[T]$. If $q = \sum_k q_k T^k \in \mathbb{C}[X][T]$, where the $q_k \in \mathbb{C}[X]$, let us set $\widetilde{q} := \sum_k \widetilde{q_k}\, T^k$, where $\widetilde{q_k} := q_k \circ F$. Since p is a vanishing polynomial of the sequence $n \mapsto F^n$, we have $\forall\, n \in \mathbb{N}$, $\sum_k p_k(X) F^{k+n}(X) = 0$. By substituting $F(X)$ to X, we get $\forall\, n \in \mathbb{N}$, $\sum_k \widetilde{p_k} F^{k+1+n} = 0$ which shows that $T\widetilde{p}(T)$ is a vanishing polynomial of the sequence $n \mapsto F^n$. If $a \,|\, b$ means that a divides b, we get $p \,|\, T\widetilde{p}$ in $\mathbb{C}(X)[T]$. Writing $p(T) = T^m q(T)$ with $q(0) \neq 0$, we get $T^m q \,|\, T^{m+1}\widetilde{q}$, so that $q \,|\, T\widetilde{q}$ and finally $q \,|\, \widetilde{q}$. Therefore, we have $p \,|\, \widetilde{p}$ and since p and $\widetilde{p}$ are monic polynomials of the same degre, we have $p = \widetilde{p}$. □

Remark. In the last proof, we need to show that each coefficient p_k of p belongs to $\mathbb{C}[X]$ in order to justify the fact that the composition $p_k \circ F$ is well defined.

Proposition 1.5. If $F \in End$ is QLF, there exist $A, B \geq 0$ such that:

$$\forall\, n \in \mathbb{N},\ \deg F^n \;\leq\; A\, n + B.$$

Proof. Let $a_0, \ldots, a_{d-1} \in \mathbb{C}[X]^F$ be such that $F^d = a_{d-1}F^{d-1} + \cdots + a_0 F^0$. Since $F^{n+d} = a_{d-1}F^{n+d-1} + \cdots + a_0 F^n$, we have $\deg F^{n+d} \;\leq \max_{0 \leq k \leq d-1} \deg a_k F^{n+k}$. If we set $d_n := \max_{0 \leq k \leq d-1} \deg F^{n+k}$, $A := \max_{0 \leq k \leq d-1} \deg a_k$ and $B := d_0$, we get $\deg F^{n+d} \;\leq A + d_n$, so that $d_{n+1} \;\leq\; A + d_n$ and $\deg F^n \;\leq\; d_n \;\leq\; A\, n + B$. □

Question. Is the converse true?

Example. Let $\mathbb{C}[Y] := \mathbb{C}[y_1, \ldots, y_m]$ and $\mathbb{C}[Z] := \mathbb{C}[z_1, \ldots, z_n]$ for $m, n \geq 1$.

Let $P := T^m - \sum_{0 \leq k \leq m-1} a_k T^k \in \mathbb{C}[Z][T]$, where the $a_k \in \mathbb{C}[Z]$. We now give a QLF endomorphism F whose minimal polynomial ν_F is equal to the least common multiple Q of P and $T - 1$.

Let $C_P := \begin{bmatrix} 0 & \ldots & 0 & a_0 \\ 1 & & 0 & a_1 \\ & \ddots & & \vdots \\ 0 & & 1 & a_{m-1} \end{bmatrix} \in M_m(\mathbb{C}[Z])$ be the Companion matrix to P.

It is well known that the minimal polynomial of C_P is equal to P. Therefore, if $F_1, \dots, F_m \in \mathbb{C}[Y, Z]$ are defined by ${}^t[F_1, \dots, F_m] = C_P.{}^t[y_1, \dots, y_m]$, it is easy to check that $F : (Y, Z) \mapsto (F_1(Y, Z), \dots, F_m(Y, Z), Z)$ is a QLF polynomial endomorphism of $\mathbb{A}^{m+n}$ satisfying $\nu_F = Q$.

Remark. Let us recall that a polynomial endomorphism $F = (F_1, \dots, F_N)$ of $\mathbb{A}^N$ is triangular if each F_L is of the form $ax_L + b$ where $a \in \mathbb{C}$ and $b \in \mathbb{C}[x_{L+1}, \dots, x_N]$. Furthermore, F is triangularisable if it is conjugate (by a polynomial automorphism) to a triangular endomorphism.

It is clear that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) in the following assertions (see [7] for (i) $\Longrightarrow$ (ii) and prop. 1.5 for (iii) $\Longrightarrow$ (iv)):

(i) F is triangularisable; (ii) F is LF; (iii) F is QLF; (iv) F is dynamically trivial.

If F is an automorphism of $\mathbb{A}^2$, it is proved in [5] that (i) and (iv) are equivalent so that the last four assertions are equivalent. However, for large values of N, these notions (applied to automorphisms) are different:

The Nagata automorphism $(x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z)$ (see [10]) is LF (see [7]) but not triangularisable (see [2]).

Using the construction explained in the last example and prop. 1.1, it is clear that the automorphism $(y, x + yz, z)$ is QLF but not LF.

If $F : \mathbb{A}^5 \to \mathbb{A}^5$, $(x, y, z, t, u) \mapsto (y, x + yz, t, z + tu, u)$, one would easily check that $\deg F^n = (n^2 - n + 4)/2$ for $n \geq 1$ so that F is dynamically trivial but not QLF by prop. 1.5.

II. MAIN THEOREM.

Here is our main result.

Theorem. Let $F \in End$. The following assertions are equivalent:

(i) for any $a \in \mathbb{A}^N$ the sequence $n \mapsto F^n(a)$ is a LRS (of the complex vector space $\mathbb{C}^N$);

(ii) there exists a non empty Zariski open subset U of $\mathbb{A}^N$ such that for any $a \in U$ the sequence $n \mapsto F^n(a)$ is a LRS;

(iii) there exists a non empty open subset U of $\mathbb{A}^N$ (for the transcendental topology) such that for any $a \in U$ the sequence $n \mapsto F^n(a)$ is a LRS;

(iv) F is QLF.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is obvious and (iv) $\Longrightarrow$ (i) is a direct consequence of prop. 1.3. Let us show that (iii) $\Longrightarrow$ (iv). If $1 \leq L \leq N$ and $\alpha \in \mathbb{N}^N$, let $\Pi_{L,\alpha}(F)$ be the coefficient of x^α of the polynomial F_L. Let $\mathcal{C} := \{\Pi_{L,\alpha}(F),\ L \in \{1, \dots, N\},\ \alpha \in \mathbb{N}^N\}$ be the set of coefficients of F and let $K := \mathbb{Q}(\mathcal{C})$ be the field extension of $\mathbb{Q}$ generated by $\mathcal{C}$.

First claim. There exists $a = (a_1, \ldots, a_N) \in U$ such that $a_1, \ldots, a_N \in \mathbb{C}$ are algebraically independant over K.

Let $R > 0$ and $u = (u_1, \ldots, u_N) \in U$ be such that:
$D := \{(z_1, \ldots, z_N) \in \mathbb{C}^N,\ 1 \leq L \leq N \Longrightarrow |z_L - u_L| < R\} \subset U$.
If we set $D_L := \{z \in \mathbb{C},\ |z - u_L| < R\}$, we have $D = D_1 \times \ldots \times D_N$. Let us construct, by finite induction on L, a complex sequence $(a_L)_{1 \leq L \leq N}$ such that for each L: $a_L \in D_L$ and a_L is transcendental over $K(a_1, \ldots, a_{L-1})$. Let us assume that $a_1, \ldots, a_{L-1}$ are already constructed and that they satisfy the wanted hypothesis. Let us note that the algebraic closure K_L of $K(a_1, \ldots, a_{L-1})$ in $\mathbb{C}$ is countable (since $K(a_1, \ldots, a_{L-1})$ is countable). Since D_L is uncountable, there exists $a_L \in D_L \setminus K_L$.

Using prop. 1.4, it is sufficient to show our

Second claim. There exists a positive integer d and rational functions $\alpha_0, \ldots, \alpha_{d-1} \in \mathbb{C}(X)$ such that $\forall\, n \in \mathbb{N},\ F^{n+d} = \alpha_{d-1} F^{n+d-1} + \cdots + \alpha_0 F^n$.

We begin to note that for each n the coefficients of F^n belong to the field K. Let us set $K' := K(a_1, \ldots, a_N)$. The sequence $(F^n(a))_{n \in \mathbb{N}}$ is a LRS of $(K')^N$ considered as a vector space over K'. If $1 \leq L \leq N$, let $\Pi_L : (K')^N \to K'$ be the L-th projection. The sequence $n \mapsto \Pi_L(F^n(a))$ being a LRS of K', its minimal polynomial μ_L belongs to $K'[T]$. Since the minimal polynomial μ of the sequence $n \mapsto F^n(a)$ satisfies $\mu = \underset{L}{lcm}\, \mu_L$, we have $\mu \in K'[T]$. Let us write $\mu = T^d - (\beta_{d-1} T^{d-1} + \cdots + \beta_0)$, where the $\beta_k \in K'$. Let $\alpha_k \in K(x_1, \ldots, x_N)$ be such that $\beta_k = \alpha_k(a_1, \ldots, a_N)$. We have:

$$\forall\, n \in \mathbb{N},\ F^{n+d}(a_1, \ldots, a_N) = \sum_{0 \leq k \leq d-1} \alpha_k(a_1, \ldots, a_N)\, F^{n+k}(a_1, \ldots, a_N)$$

and since $a_1, \ldots, a_N$ are algebraically independant over K, we obtain:

$$\forall\, n \in \mathbb{N},\ F^{n+d}(X) = \sum_{0 \leq k \leq d-1} \alpha_k(X)\, F^{n+k}(X). \qquad \square$$

Remarks. 1. Let us recall that the rank of a LRS u is the degree of its minimal polynomial. If u is a complex sequence, its Hankel matrix is defined by

$$H(u) := \begin{bmatrix} u_0 & u_1 & \ldots & u_n & \ldots \\ u_1 & u_2 & \ldots & u_{n+1} & \ldots \\ \vdots & \vdots & & \vdots & \\ u_n & u_{n+1} & \ldots & u_{2n} & \ldots \\ \vdots & \vdots & & \vdots & \end{bmatrix} \text{ and we have:}$$

$\operatorname{rk} u \leq m \Longleftrightarrow$ all the $k \times k$ minors of $H(u)$ are zero for $k \geq m + 1$.

If $F \in End$ is QLF, let $\varphi_F : \mathbb{A}^N \to \mathbb{N}$ be the map associating to $a \in \mathbb{A}^N$ the rank of the LRS $n \mapsto F^n(a)$. Using the previous point, is is easy to show that φ_F is lower semicontinuous. This means that for each $m \geq 0$, the set $F_m := \{a \in \mathbb{A}^N,\ \varphi_F(a) \leq m\}$ is a (Zariski) closed subset of $\mathbb{A}^N$.

2. The proof of the last theorem shows us that $\deg \nu_F = \max_{a \in \mathbb{A}^N} \varphi_F(a)$. However, let us

show that φ_F is upper bounded by using the semicontinuity. The equality $\mathbb{A}^N = \bigcup_{n\geq 0} F_n$ implies that $\mathbb{A}^N = F_n$ for some $n \geq 0$. Otherwise, the $U_n := \mathbb{A}^N \setminus F_n$ would be dense open subsets of $\mathbb{A}^N$ satisfying $\bigcap_{n\geq 0} U_n = \emptyset$ and this would contradict the Baire property.

III. CRITERIA FOR INVERTIBILITY.

Let us denote by $I := (x_1, \ldots, x_N)$ the identity morphism of $\mathbb{A}^N$.

Proposition 3.1. If $F \in End$ is QLF, then F is an automorphism if and only if $\nu_F(0) \in \mathbb{C}^*$.

Proof. Let us write $\nu_F = \sum_{0\leq k\leq n} a_k T^k$, where the $a_k \in \mathbb{C}[X]$ and $a_n = 1$. If F is an automorphism, we cannot have $a_0 = 0$, because otherwise $p(T) := \nu_F(T)\, T^{-1} \in \mathbb{C}[X]^F[T]$ and $p(F) \circ F = 0$. Since F is onto, this would imply $p(F) = 0$ contradicting the definition of ν_F. One would easily check that $\nu_{F^{-1}} = a_0^{-1}\, T^n\, \nu_F(T^{-1})$. By prop. 1.3, each coefficient of $\nu_{F^{-1}}$ belongs to $\mathbb{C}[X]$. In particular, the constant coefficient a_0^{-1}. Since a_0 and $a_0^{-1} \in \mathbb{C}[X]$, a_0 is an invertible element of $\mathbb{C}[X]$ so that $a_0 \in \mathbb{C}^*$. Conversely, if $a_0 \in \mathbb{C}^*$, then $q(T) := \frac{a_0 - \nu_F(T)}{a_0 T} \in \mathbb{C}[X]^F[T]$ satisfies $q(T)T \equiv 1 \mod \nu_F(T)$, so that $q(F) \circ F = I$ and F is an automorphism. □

The Jacobian determinant of an endomorphism F will be denoted by $\operatorname{Jac} F$.

Proposition 3.2. If $F \in End$ is QLF, then the Jacobian conjecture holds for F, i.e. F is an automorphism if and only if $\operatorname{Jac} F \in \mathbb{C}^*$.

Proof. If F is an automorphism it is well known and obvious that $\operatorname{Jac} F \in \mathbb{C}^*$. Conversely, if $F \in End$ is QLF and satisfies $\operatorname{Jac} F \in \mathbb{C}^*$, let us show that F is an automorphism. If we write $\nu_F = \sum_{0\leq k\leq n} a_k T^k$, where the $a_k \in \mathbb{C}[X]$ and $a_n = 1$, it is sufficient to show that $a_0 \in \mathbb{C}^*$. First and foremost, we cannot have $a_0 = 0$. Indeed, otherwise, we would have $p(F) \circ F = 0$, where $p := \nu_F(T)\, T^{-1} \in \mathbb{C}[X]^F[T]$. If $r \in \mathbb{C}[X]$ denotes a nonzero coordinate of $p(F)$, we would get $r(F) = 0$, showing that $F_1, \ldots, F_N$ are algebraically dependant over $\mathbb{C}$. This is well known to be equivalent to $\operatorname{Jac} F = 0$ (see [11]) which is impossible. If we set $q(T) := \frac{a_0 - \nu_F(T)}{a_0 T} \in \mathbb{C}(X)^F[T]$, then $q(T)T \equiv 1 \mod \nu_F(T)$, so that $q(F) \circ F = I$. This shows that F is a birational automorphism. Since $\operatorname{Jac} F \in \mathbb{C}^*$, this is well known to imply that F is an automorphism (see th. 2.1 of [1], cor. 1.1.35 of [4] or [8]). □

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