Polynomial Composition Rigidity and Plane Polynomial Automorphisms.

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Abstract.

Let $\mathcal{G}$ be the group of polynomial automorphisms of the complex affine plane. On one hand, $\mathcal{G}$ can be endowed with the structure of an infinite dimensional algebraic group (see [26]) and on the other hand there is a partition of $\mathcal{G}$ according to the multidegree (see [5]). Let $\mathcal{G}_d$ denote the set of automorphisms whose multidegree is equal to d. In this paper, we state a conjecture dealing with polynomial composition in one indeterminate and we prove it in a few particular cases. This conjecture allows us to describe the closure of $\mathcal{G}_d$ when d is of length 2.

Keywords.

Affine space, Polynomial automorphisms.

MSC.

14R10.

INTRODUCTION.

Let K be a field of characteritic 0. It is well known that the following statement of Abhyankar-Moh-Suzuki (see [1, 28]) dealing with polynomials in one indeterminate easily implies Jung's theorem (see [14]) asserting that any polynomial automorphism of the affine plane $\mathbb{A}^2_K$ is a composition of affine and triangular automorphisms.

Abhyankar-Moh-Suzuki Theorem. If $a, b \in K[X]$ are such that $K[a,b] = K[X]$, then $\deg a$ divides $\deg b$ or $\deg b$ divides $\deg a$.

In the same vein, we think that the following statement dealing with polynomials in one indeterminate is true for any $m, n \geq 1$ and we will show that it has some application in the theory of plane polynomial automorphisms. Using Gröbner basis, we have checked it when $m \leq 5$ and $n \leq 8$.

R(m,n). Let $a = X(1+a_1X+\ldots+a_mX^m)$, $b = X(1+b_1X+\ldots+b_nX^n) \in \mathbb{C}[X]$, where the $a_i, b_j \in \mathbb{C}$. Let us write $a \circ b = X(1+c_1X+\ldots c_NX^N)$, where $N = (m+1)(n+1)-1$

and the $c_k \in \mathbb{C}$. If $c_1 = \ldots = c_{m+n} = 0$, then $a = b = X$.

Let us begin to explain why this may be interpreted as a rigidity statement. Considering a_i, b_i as indeterminates of degree i allows us to see each c_i as a homogeneous polynomial of degree i in $\mathbb{C}[a_1, \ldots, a_m, b_1, \ldots, b_n]$. The hypothesis $R(m,n)$ means that $c_1, \ldots, c_{m+n}$ is a regular system of parameters of $\mathbb{C}[a_1, \ldots, a_m, b_1, \ldots, b_n]$. For the notion of a regular system of parameters, see any book of commutative algebra (for example [4]) or §2.2 below. Therefore, $R(m,n)$ is still equivalent to asserting that the polynomial endomorphism of $\mathbb{A}_{\mathbb{C}}^{m+n}$ sending $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ to $(c_1, \ldots, c_{m+n})$ is quasi-finite (i.e. each of its fiber is a finite set). In some sense, this means that polynomial composition is rigid.

In this connection, let us recall the famous result of Ritt on polynomial composition (see [22]). A polynomial $a \in \mathbb{C}[X]$ of degree ≥ 2 is prime if the relation $a = b \circ c$ implies that b or c has degree 1. If $\deg f \geq 2$, it is clear that f admits a decomposition $f = f_1 \circ \ldots \circ f_r$ into prime polynomials. Let us note that $a \circ b = c \circ d$ in the three following cases:

(i) a, b are any polynomials and $c = a \circ l$, $d = l^{-1} \circ b$, where l has degree 1 and l^{-1} is its inverse for the composition;

(ii) $a = d = t_m$ and $b = c = t_n$, where $t_m(X) = \cos(m \arccos X)$ is the m-th Chebyshev polynomial;

(iii) $a = d = X^m$, $b = X^n p(X^m)$ and $c = X^n p(X)^m$, where p is any polynomial, and the converse situation, i.e. $b = c = X^m$, $d = X^n p(X^m)$ and $a = X^n p(X)^m$.

If $f = f_1 \circ \ldots \circ f_r$ is a prime decomposition, then, by replacing an adjacent pair $(f_i, f_{i+1}) = (a, b)$ by (c, d) where a, b, c, d are as in (i-iii) above, we obtain a new prime decomposition. This process is called an elementary transformation.

Ritt's Theorem. The prime decomposition is unique modulo elementary transformations, i.e. if we are given two prime decompositions of some polynomial, we can pass from one to the other by applying elementary transformations.

Many problems related to polynomial composition (and iteration!) are intricate. For example, the famous Mandelbrot set is defined as the set of complex c-values for which the orbit of 0 under iteration of $p(X) = X^2 + c$ remains bounded! In fact, $R(m,n)$ is related to the next hypothesis which perhaps looks more attractive.

R(m). Let $a = X(1 + a_1 X + \ldots + a_m X^m) \in \mathbb{C}[X]$ and let $a^{-1} \in \mathbb{C}[[X]]$ be its formal inverse for the composition. If m consecutive coefficients of a^{-1} vanish, then $a = X$.

If $a^{-1} = X\left(1 + \sum_{n \geq 1} b_n X^n\right)$, the vanishing conditions means that there exists an integer n such that $b_{n+k} = 0$ for $0 \leq k < m$. With words, the conjecture $R(m)$ means that the inverse for the composition of a non trivial polynomial is badly approximated by polynomials. Let us remark the analogy with Heisenberg's uncertainty principle in quantum mechanics asserting that one cannot reduce arbitrarily the uncertainty as to the position and the momentum of a free particle (see [29] and [19] for mathematical statements). In fact, many mathematical results contain a close idea. For example, rationals are badly approximated by rationnals: if α is any real number, Hurwitz has proved that there are infinitely many rationals $\dfrac{p}{q}$ such that $\left|\alpha - \dfrac{p}{q}\right| < \dfrac{1}{\sqrt{5}q^2}$ if and only if α is irrational (see [13]). Algebraic numbers are also badly approximated by rationals. Roth has proved that for algebraic α and $\varepsilon > 0$, there exist only finitely many rationals $\dfrac{p}{q}$ such that $\left|\alpha - \dfrac{p}{q}\right| < \dfrac{1}{q^{2+\varepsilon}}$ (see [23]). On the converse, this idea was first used by Liouville to construct transcental numbers (see [17, 18]). We will show below (see lemma 1.2) that $R(m)$ holds if and only if $R(m, n)$ holds for any n. Coming back to our initial formulation $R(m, n)$, our first result is the following:

Theorem A. If m or $n \leq 2$, then $R(m, n)$ is true.

To give an application to plane polynomial automorphisms, we need some notations. A polynomial endomorphism of $\mathbb{A}^2_K$ will be identified with its sequence $f = (f_1, f_2)$ of coordinate functions $f_j \in K[X, Y]$. We define its degree by deg $f = \max_{1 \leq j \leq 2}$ deg f_j.

The space $\mathcal{E} := \mathbb{C}[X, Y]^2$ of polynomial endomorphisms of $\mathbb{A}^2_{\mathbb{C}}$ is naturally an infinite dimensional algebraic variety (see [25, 26] for the definition). This roughly means that $\mathcal{E}_{\leq m} := \{f \in \mathcal{E},\ \deg f \leq m\}$ is a (finite dimensional) algebraic variety for any $m \geq 1$, which comes out from the fact that it is an affine space. If $Z \subseteq \mathcal{E}$, we set $Z_{\leq m} := Z \cap \mathcal{E}_{\leq m}$. The space $\mathcal{E}$ is endowed with the topology of the inductive limit, in which Z is closed (resp. open, resp. locally closed) if and only if $Z_{\leq m}$ is closed (resp. open, resp. locally closed) in $\mathcal{E}_{\leq m}$ for any m.

Let $\mathcal{G}$ be the group of polynomial automorphisms of $\mathbb{A}^2_{\mathbb{C}}$. Since $\mathcal{G}$ is locally closed in $\mathcal{E}$ (see [2, 25, 26]), it is naturally an infinite dimensional algebraic variety.

Let $\mathcal{A} := \{(aX+bY+c, dX+eY+f),\ a, b, c, d, e, f \in \mathbb{C},\ ae-bd \neq 0\}$ be the subgroup of affine automorphisms and $\mathcal{B} := \{(aX + p(Y), bY + c),\ a, b, c \in \mathbb{C}, p \in \mathbb{C}[Y],\ ab \neq 0\}$ be the subgroup of triangular automorphisms ($\mathcal{B}$ may be viewed as a Borel subgroup of $\mathcal{G}$). If $f \in \mathcal{G}$, by [14, 16, 21] one can write $f = \alpha_1.\beta_1.\ldots.\alpha_k.\beta_k.\alpha_{k+1}$ where the α_j's (resp. β_j's) belong to $\mathcal{A}$ (resp. $\mathcal{B}$). By contracting such an expression, one might as well suppose that it is reduced, i.e. $\forall j,\ \beta_j \notin \mathcal{A}$ and $\forall j,\ 2 \leq j \leq k,\ \alpha_j \notin \mathcal{B}$. It follows from the amalgamated structure of $\mathcal{G}$ that if $f = \alpha'_1.\beta'_1.\ldots.\alpha'_l.\beta'_l.\alpha'_{l+1}$ is another reduced expression of f, then $k = l$ and there exist $(\gamma_j)_{1 \leq j \leq k}$, $(\delta_j)_{1 \leq j \leq k}$ in $\mathcal{A} \cap \mathcal{B}$ such that $\alpha'_1 = \alpha_1.\gamma_1^{-1}$, $\alpha'_j = \delta_{j-1}.\alpha_j.\gamma_j^{-1}$ (for $2 \leq j \leq k$), $\alpha'_{k+1} = \delta_k.\alpha_{k+1}$ and $\beta'_j = \gamma_j.\beta_j.\delta_j^{-1}$ (for $1 \leq j \leq k$). Following [5] (resp. [7]), we define the multidegree (resp. length) of f by $d(f) :=$ (deg $\beta_1, \ldots,$ deg β_k) (resp. $l(f) = k$). In fact, these notions of multidegree and length could be defined in the same way for a polynomial automorphism of $\mathbb{A}^2_K$, where K is any field. We recall that degree and multidegree are related by $\deg f =$ deg $\beta_1 \times \ldots \times$ deg β_k (see [30, 5]). Let $\mathcal{D}$ be the set of multidegrees, i.e. of finite sequences of integers ≥ 2 (including the empty sequence). If $d = (d_1, \ldots, d_l) \in \mathcal{D}$, $\mathcal{G}_d$

will denote the set of automorphisms whose multidegree is equal to d. By [5, 9], $\mathcal{G}_d$ is an irreducible smooth, locally closed subset of $\mathcal{G}$ of dimension $d_1 + \ldots + d_l + 6$. Let us note that $\mathcal{G}_d \subseteq \mathcal{G}_{\leq n}$ as soon as $n \geq d_1 \ldots d_l$ and that we have a partition of $\mathcal{G}$ as a disjoint union $\mathcal{G} = \coprod_{d \in \mathcal{D}} \mathcal{G}_d$. What can be said on the closure of $\mathcal{G}_d$?

By [7], the length of an automorphism is lower semicontinuous. Therefore, any element of $\overline{\mathcal{G}}_d$ has length $\leq l$. The simplest non trivial related example is given by the Nagata automorphism $N := (X - 2YW - ZW^2, Y + ZW, Z)$, where $W = XZ + Y^2$(see [21]). This automorphism of $\mathbb{A}^3_{\mathbb{C}}$ can be seen as an automorphism of $\mathbb{A}^2_{\mathbb{C}[Z]}$ inducing as well a morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathcal{G}$, $z \mapsto N_z$. If $z \neq 0$, the factorization $N_z = (X - z^{-1}Y^2, Y) \circ (X, Y + z^2 X) \circ (X + z^{-1}Y^2, Y)$ shows us that N_z has multidegree $(2, 2)$. If $z = 0$, $N_0 = (X - 2Y^3, Y)$ so that N_0 has multidegree (3). This yields us $\mathcal{G}_3 \cap \overline{\mathcal{G}}_{(2,2)} \neq \emptyset$. Inspired by the analysis of similar examples, we hoped (erroneously) that the equality $\overline{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$ might always be true, where $\preceq$ is defined in the following way:

Definition. $\preceq$ is the partial order on $\mathcal{D}$ induced by the relations:

(i) $\emptyset \preceq d$ (for any d);
(ii) $(d_1, \ldots, d_k) \preceq (e_1, \ldots, e_k)$ when $d_j \leq e_j$ for any j;
(iii) $(d_1, \ldots, d_{j-1}, d_j + d_{j+1} - 1, d_{j+2}, \ldots, d_k) \preceq (d_1, \ldots, d_k)$ when $1 \leq j \leq k - 1$.

However, by [3], there cannot exist any partial order $\sqsubseteq$ such that $\overline{\mathcal{G}}_d = \bigcup_{e \sqsubseteq d} \mathcal{G}_e$ when $d = (11, 3, 3)$! Actually, it is proved there that $\mathcal{G}_{(19)} \cap \overline{\mathcal{G}}_{(11,3,3)} \neq \emptyset$ and for grounds of dimension we cannot have $\mathcal{G}_{(19)} \subseteq \overline{\mathcal{G}}_{(11,3,3)}$. As a matter of fact reality is often complex. We now think that the equality $\overline{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$ is actually true in length ≤ 2 ! It has been proved in length ≤ 1 in [6] and this paper settles the length 2 case modulo $R(m, n)$:

Theorem B. If $R(m, n)$ is true and $d = (m + 1, n + 1)$, then $\overline{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$.

Remark. If $\overline{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$ holds in length ≤ 2, we get $\overline{\mathcal{G}}_d \supseteq \bigcup_{e \preceq d} \mathcal{G}_e$ for any d.

Theorems A and B directly imply:

Theorem C. If $d = (d_1, d_2) \in \mathcal{D}$ with d_1 or $d_2 \leq 3$, then $\overline{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$.

As a funny and easy consequence of the last outcome, we will show:

Theorem D. Any closed subgroup of $\mathcal{G}$ containing the affine group and an automorphism of length 1 is equal to $\mathcal{G}$.

Remarks. 1. Let us note that any subgroup of $\mathcal{G}$ strictly containing the affine group is dense in $\mathcal{G}$ for the Krull topology (see theorem A of [8]). This means that any element of $\mathcal{G}$ can be approximated at the origin and at any order by an element of this subgroup. Furthermore, by a result of Shafarevich, any closed subgroup of $\mathcal{G}$ whose Lie algebra equals the one of $\mathcal{G}$ is equal to $\mathcal{G}$ (see theorem 1 of [26] and [27]). However, one cannot obtain from these results the statement (stronger than theorem D) that any closed subgroup of $\mathcal{G}$ strictly containing the affine group is equal to $\mathcal{G}$. Indeed, even if such a subgroup is dense in $\mathcal{G}$ for the Krull topology, it is not clear (for us) that its Lie algebra equals the one of $\mathcal{G}$.

2. We recall that any closed subgroup of $\mathcal{G}$ which is a finite dimensional algebraic variety is conjugated to either a subgroup of $\mathcal{A}$ or $\mathcal{B}$ (see theorem 8 of [25], theorem 4.3 of [15] or theorem 7 of [10]).

Question. Does there exist a non trivial closed subgroup of $\mathcal{G}$ containing the affine group?

The paper is divided into four sections. In section 1, we study the rigidity's hypothesis $R(m, n)$ and prove theorem A. In section 2, we give some preliminary results to be used in section 3 where we prove theorem B. In section 4, we prove theorem D.

I. THE RIGIDITY'S HYPOTHESIS.

In subsection 1, we give some generalities. In subsection 2, we prove theorem A and in subsection 3, we study the hypothesis $R(3)$.

1. Generalities.

We begin to show that m and n play the same role in $R(m, n)$:

Lemma 1.1. $R(m, n)$ is equivalent to $R(n, m)$.

Proof. Let $\mathrm{val} : \mathbb{C}((X)) \to \mathbb{Z} \cup \{+\infty\}$ be the valuation associated to the discrete valuation ring $\mathbb{C}[[X]]$.

$R(m, n)$ means that if $a = X(1+a_1X+\ldots+a_mX^m)$ and $b = X(1+b_1X+\ldots+b_nX^n)$ are such that $\mathrm{val}(a \circ b - X) \geq m+n+2$, then $a = b = X$. Therefore, it is enough to note that $\mathrm{val}(a \circ b - X) = \mathrm{val}(b \circ a - X)$.

Indeed, if $p := a \circ b - X$, $q := b \circ a - X$ and $a^{-1}(X) \in \mathbb{C}[[X]]$ is the formal inverse of a for the composition, we have $p = a \circ q \circ a^{-1}$, so that $\mathrm{val}\, p = \mathrm{val}\, q$. We have used the fact that $\mathrm{val}(u \circ v) = \mathrm{val}\, u \times \mathrm{val}\, v$ when $u, v \in X\mathbb{C}[[X]]$. $\square$

As announced, we establish the link between $R(m,n)$ and $R(m)$:

Lemma 1.2. If $m \geq 1$, the two following assertions are equivalent:
(i) $R(m)$ holds;
(ii) $R(m,n)$ holds for any $n \geq 1$.

Proof. If $a = X(1 + a_1X + \ldots + a_mX^m)$ and $b = X(1 + b_1X + \ldots + b_nX^n)$, we have $\mathrm{val}(a \circ b - X) = \mathrm{val}(b - a^{-1})$. □

In the next computation, we found it convenient to express $a = X(1 + a_1X + \ldots + a_mX^m)$ as $a = X(1 - \lambda_1X)\ldots(1 - \lambda_mX)$.

Lemma 1.3. If $a = X(1 - \lambda_1X)\ldots(1 - \lambda_mX)$, we have $a^{-1} = X\left(1 + \sum_{n\geq 1} \frac{u_n}{n+1} X^n\right)$, where $u_n = \sum_{j_1+\ldots+j_m=n} \binom{n+j_1}{n} \cdots \binom{n+j_m}{n} \lambda_1^{j_1} \ldots \lambda_m^{j_m}$.

Proof. If $\oint$ denotes integration over a little circle around the origin, it comes out (by Lagrange formula):

$$u_n = \frac{n+1}{2\pi i} \oint \frac{a^{-1}(X)}{X^{n+2}}\, dX = \frac{n+1}{2\pi i} \oint \frac{Y}{a^{n+2}(Y)}\, a'(Y) dY \qquad \text{by setting } X = a(Y);$$

$$= \frac{1}{2\pi i} \oint \frac{dY}{a^{n+1}(Y)} \qquad \text{by integrating by parts.}$$

Therefore, u_n is the X^n-coefficient of $\prod_{1\leq k\leq m} \frac{1}{(1-\lambda_k X)^{n+1}} \in \mathbb{C}[[X]]$ and we conclude thanks to the Taylor series expansion $\frac{1}{(1-X)^{n+1}} = \sum_{j\geq 0} \binom{n+j}{n} X^j$. □

The assertion $R(m)$ means that if m consecutive u_n vanish, then the λ_i also. It is sufficient to show that $u_k = 0$ when k is big enough, because in this case a^{-1} is a polynomial and the relation $X = a \circ a^{-1}$ shows us that $a = X$ (by taking the degree). We will adopt this point of view in the next subsections.

2. Proof of theorem A.

According to lemmas 1.1 and 1.2, it is sufficient to prove $R(m)$ for $m = 1, 2$. We take the notations of lemma 1.3 and proceed as explained after this lemma.

Proof of *R(1)*. If $u_n = \binom{2n}{n}\lambda_1^n$, we want to show that the equality $u_k = 0$ (for some k) implies $\lambda_1 = 0$.

This is obvious. $\square$

Proof of *R(2)*. If $u_n = \sum_{i+j=n} \binom{n+i}{n}\binom{n+j}{n}\lambda_1^i\lambda_2^j$, we want to show that the equality $u_k = u_{k+1} = 0$ (for some k) implies $\lambda_1 = \lambda_2 = 0$.

This will come out from the following linear recurrence relation which we have found by using a computer but which is easy to check by hand (exercise!):

$$n(n-1)(\lambda_1-\lambda_2)^2u_n + (n-1)(2n-1)(\lambda_1+\lambda_2)(\lambda_1-2\lambda_2)(\lambda_2-2\lambda_1)u_{n-1} \\ -3(3n-4)(3n-2)\lambda_1^2\lambda_2^2u_{n-2} = 0.$$

Let us begin to show by contradiction that $\lambda_1 = \lambda_2$. Otherwise, the above relation shows us that for any n the following implication holds: $(u_{n-2} = u_{n-1} = 0 \Longrightarrow u_n = 0)$. Therefore, $u_n = 0$ for $n \geq k$ and we have seen that this implies $\lambda_1 = \lambda_2 = 0$. A contradiction.

Let us now set $\lambda := \lambda_1 = \lambda_2$. The above relation gives us:

$$2(n-1)(2n-1)\lambda^3u_{n-1} - 3(3n-4)(3n-2)\lambda^4u_{n-2} = 0.$$

Let us show by contradiction that $\lambda = 0$. Otherwise, the last relation shows us that for any n the following implication holds: $(u_{n-2} = 0 \Longrightarrow u_{n-1} = 0)$. We still get $u_n = 0$ for $n \geq k$, so that $\lambda = 0$. A contradiction. $\square$

3. The hypothesis $R(3)$.

Let $u_n := \sum_{i+j+k=n} \binom{n+i}{n}\binom{n+j}{n}\binom{n+k}{n}\lambda_1^i\lambda_2^j\lambda_3^k \in \mathbb{Z}[\lambda_1,\ \lambda_2,\ \lambda_3]$.

By lemma 1.3, the hypothesis $R(3)$ is equivalent to saying that u_k, u_{k+1}, u_{k+2} is a regular system of parameters of $\mathbb{C}[\lambda_1,\ \lambda_2,\ \lambda_3]$ (for any $k \geq 1$).

Before giving the linear recurrence relation satisfied by the u_n, we need some notations. If $\mu = (\mu_1, \mu_2, \mu_3)$ where μ_i are integers satisfying $\mu_1 \geq \mu_2 \geq \mu_3 \geq 0$, we define $m_\mu \in \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3]$ by $m_\mu := \sum \lambda_1^{\nu_1}\lambda_2^{\nu_2}\lambda_3^{\nu_3}$ where (ν_1, ν_2, ν_3) describes all distinct permutations of the triple (μ_1, μ_2, μ_3). We will identify (μ_1, μ_2) and $(\mu_1, \mu_2, 0)$ as well as (μ_1) and $(\mu_1, 0, 0)$. Hence $m_{(1)} = \lambda_1 + \lambda_2 + \lambda_3$, $m_{(1,1)} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$, $m_{(1,1,1)} = \lambda_1\lambda_2\lambda_3$ and $m_{(3,2,1)} = \lambda_1^3\lambda_2^2\lambda_3 + \lambda_1^3\lambda_2\lambda_3^2 + \lambda_1^2\lambda_2^3\lambda_3 + \lambda_1\lambda_2^3\lambda_3^2 + \lambda_1^2\lambda_2\lambda_3^3 + \lambda_1\lambda_2^2\lambda_3^3$.

We set $\Delta = (\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2$ and define the M_i, N_i for $1 \leq i \leq 5$ by:

$M_1 = 9m_{(4,2)} - 14m_{(4,1,1)} - 9m_{(3,3)} + 3m_{(3,2,1)} - 3m_{(2,2,2)}$;
$M_2 = 5m_{(4,2)} - 8m_{(4,1,1)} - 5m_{(3,3)} + 2m_{(3,2,1)} - 3m_{(2,2,2)}$;
$M_3 = 39m_{(4,2)} - 62m_{(4,1,1)} - 39m_{(3,3)} + 15m_{(3,2,1)} - 21m_{(2,2,2)}$;
$M_4 = 33m_{(4,2)} - 52m_{(4,1,1)} - 33m_{(3,3)} + 12m_{(3,2,1)} - 15m_{(2,2,2)}$;

$M_5 = 6m_{(4,2)} - 10m_{(4,1,1)} - 6m_{(3,3)} + 3m_{(3,2,1)} - 6m_{(2,2,2)}$;

$N_1 = -2m_{(5,2)} + 4m_{(5,1,1)} + 3m_{(4,3)} - 3m_{(4,2,1)} - 8m_{(3,3,1)} + 8m_{(3,2,2)}$;
$N_2 = 10m_{(9,4)} - 36m_{(9,3,1)} + 52m_{(9,2,2)} - 25m_{(8,5)} + 63m_{(8,4,1)} - 38m_{(8,3,2)} + 10m_{(7,6)} + 30m_{(7,5,1)} - 146m_{(7,4,2)} + 216m_{(7,3,3)} - 60m_{(6,6,1)} + 70m_{(6,5,2)} - 32m_{(6,4,3)} - 60m_{(5,5,3)} + 40m_{(5,4,4)}$;
$N_3 = -27m_{(4,4)} + 36m_{(4,3,1)} - 2m_{(4,2,2)} - 52m_{(3,3,2)}$;
$N_4 = -342m_{(8,6)} + 1006m_{(8,5,1)} - 1110m_{(8,4,2)} + 972m_{(8,3,3)} + 342m_{(7,7)} - 141m_{(7,6,1)} - 1301m_{(7,5,2)} + 900m_{(7,4,3)} + 1178m_{(6,6,2)} - 15m_{(6,5,3)} - 724m_{(6,4,4)} + 238m_{(5,5,4)}$;
$N_5 = 10m_{(8,6)} - 30m_{(8,5,1)} + 34m_{(8,4,2)} - 30m_{(8,3,3)} - 10m_{(7,7)} + 5m_{(7,6,1)} + 37m_{(7,5,2)} - 27m_{(7,4,3)} - 38m_{(6,6,2)} + 5m_{(6,5,3)} + 20m_{(6,4,4)} - 10m_{(5,5,4)}$.

If we now set
$A_n = n(n-1)(n-2)(M_1 n - 3M_2)\Delta$,
$B_n = (n-1)(n-2)[(2M_1 n^2 - M_3 n)N_1 - 3N_2]$,
$C_n = (n-2)[(M_1 n^3 - M_4 n^2)N_3 + 3nN_4 + 36N_5]$,
$D_n = 8(2n-3)(4n-7)(4n-9)(M_1 n - M_5)m^3_{(1,1,1)}$,
one can show (using a computer!) that for $n \geq 4$, we have:

$$A_n u_n + B_n u_{n-1} + C_n u_{n-2} + D_n u_{n-3} = 0.$$

The difference with the recurrence relations obtained for $R(1)$ and $R(2)$ is that the coefficient $M_1 n - 3M_2$ of A_n depends on n and the λ_i together. Therefore, if we assume that $u_k = u_{k+1} = u_{k+2} = 0$, we do not succed to show that $u_{k+3} = 0$! However, this could probably be done by a closer analysis.

II. PRELIMINARY RESULTS.

In subsection 1, we give a valuative criterion to characterize the elements of $\overline{f(V)}$ where $f : V \to W$ is a morphism of complex algebraic varieties. The only valuation ring we need is the ring of complex formal power series.

In subsection 2, after recalling equivalent definitions of a regular system of parameters, we give two results related with formal power series.

Finally, in subsection 3, we make some technical definitions which will allow us (in the next section) to prove theorem B by applying the results of subsections 1 and 2.

1. Valuative criterion.

Even if the criterion below sounds familiar (see for example [20], chap. 2, §1, pp 52-54 or [11], §7), we have given a proof of it in [9].

Let $\mathbb{C}[[T]]$ be the algebra of complex formal power series and let $\mathbb{C}((T))$ be its quotient field. If V is a complex algebraic variety and A a complex algebra, $V(A)$ will denote the points of V with values in A, i.e. the set of morphisms $\operatorname{Spec} A \to V$. If v is a closed point of V and $\varphi \in V\Big(\mathbb{C}((T))\Big)$, we will write $v = \lim\limits_{T\to 0} \varphi(T)$ when:

(i) the point $\varphi : \mathrm{Spec}\, \mathbb{C}((T)) \to V$ is a composition $\mathrm{Spec}\, \mathbb{C}((T)) \to \mathrm{Spec}\, \mathbb{C}[[T]] \to V$;
(ii) v is the point $\mathrm{Spec}\, \mathbb{C} \to \mathrm{Spec}\, \mathbb{C}[[T]] \to V$.

For example, if $V = \mathbb{A}^1_{\mathbb{C}}$ and $\varphi \in V\Big(\mathbb{C}((T))\Big) = \mathbb{C}((T))$, we will write $v = \lim_{T\to 0} \varphi(T)$ when $\varphi \in \mathbb{C}[[T]]$ and $v = \varphi(0)$.

Valuative criterion. Let $f : V \to W$ be a morphism of complex algebraic varieties and let w be a closed point of W. The two following assertions are equivalent:
(i) $w \in \overline{f(V)}$;
(ii) $w = \lim_{T\to 0} f(\varphi(T))$ for some $\varphi \in V\Big(\mathbb{C}((T))\Big)$.

Remark. Note the analogy with the metric case where $w \in \overline{f(V)}$ if and only if there exists a sequence $(v_n)_{n\geq 1}$ of V such that $w = \lim_{n\to+\infty} f(v_n)$.

Using this criterion, it is easy to deduce the following result (see [9], corollary 1.1):

Corollary 2.1. If $d \in \mathcal{D}$ and $f \in \mathcal{G}$, the following assertions are equivalent:
(i) $f \in \overline{\mathcal{G}}_d$;
(ii) $f = \lim_{T\to 0} g_T$ for some $g \in \mathcal{G}_d\Big(\mathbb{C}((T))\Big)$.

Remark. Since $\mathcal{G}_d$ is locally closed in $\mathcal{G}$ (see [9]), there is a natural identification between $\mathcal{G}_d(K)$ and the set of automorphisms of $\mathbb{A}^2_K$ whose multidegree is equal to d, for any field K containing $\mathbb{C}$.

2. Regular system of parameters and formal power series.

Let K be an algebraically closed field and let $\mathbb{A}^r_K$ be the affine r-space over K. Let us grade the polynomial algebra $R = K[z_1, \ldots, z_r]$ by assigning each z_k to be homogeneous of some strictly positive degree (depending on k). For each $m \geq 0$, we agree that R_m is the set of m-homogeneous polynomials of R. If $p = (p_1, \ldots, p_r) \in R^r$, let $\phi_p : \mathbb{A}^r_K \to \mathbb{A}^r_K$ be the morphism of algebraic varieties defined by $\phi_p(z) = (p_1(z), \ldots, p_r(z))$ for $z \in \mathbb{A}^r_K$. Let also I_p be the ideal of R generated by $p_1, \ldots, p_r$. If each p_k is homogeneous of some strictly positive degree (depending on k), we recall the following criterion and its corollary.

Regularity Criterion. The r-uple p is a **regular system of parameters** of R if the following equivalent assertions are satisfied:
(i) $(\phi_p)^{-1}(0) = \{0\}$;
(ii) ϕ_p is a finite morphism;
(iii) ϕ_p is a quasi-finite morphism;
(iv) ϕ_p is a proper morphism;
(v) ϕ_p is a flat morphism;

(vi) $\dim_K R/I_p < +\infty$;
(vii) if $d > \max_k \deg p_k$, then for big enough l we have $R_{dl} \subseteq I^l$.

Corollary 2.2. If $p = (p_1, \ldots, p_r)$ is a regular system of parameters of $K[z_1, \ldots, z_r]$, the map $\phi_p : \mathbb{A}^r_K \to \mathbb{A}^r_K$ is surjective.

Proof. Since ϕ_p is proper (resp. flat), its image is closed (resp. open). □

Let val : $\mathbb{C}((T)) \to \mathbb{Z} \cup \{+\infty\}$ be the valuation associated to the discrete valuation ring $\mathbb{C}[[T]]$.

Lemma 2.1. If $(p_1, \ldots, p_r)$ is a regular system of parameters of $R := \mathbb{C}[z_1, \ldots, z_r]$ and $q \in R$ is homogeneous with $\deg q > \max_k \deg p_k$, then for any $b \in \mathbb{A}^r_{\mathbb{C}[[T]]}$ satisfying $b(0) = 0$, we have:

$$\operatorname{val} q(b) \ \geq \ \min_k \ \operatorname{val} p_k(b) \ + \ 1.$$

Proof. By (vii) of the regularity criterion, there exists $l \geq 1$ such that $q^l \in (p_1, \ldots, p_r)^l$. For any $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, let us set $p^\alpha = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_r$.

If $A := \{\alpha \in \mathbb{N}^r, |\alpha| = l\}$, we can write $q^l = \sum_{\alpha \in A} s_\alpha \, p^\alpha$ $(s_\alpha \in R)$. Furthermore, we may assume that $s_\alpha \in R_{d_\alpha}$, where $d_\alpha := \deg q^l - \deg p^\alpha \geq l(\deg q - \max_k \deg p_k) \geq l \geq 1$. Evaluating at b and taking the valuation, we get: $l \operatorname{val} q(b) \geq \min_{\alpha \in A} \operatorname{val} s_\alpha(b) p^\alpha(b)$. But $\operatorname{val} s_\alpha(b) \geq 1$ and $\operatorname{val} p^\alpha(b) \geq l \ \min_k \ \operatorname{val} p_k(b)$, so that $l \operatorname{val} q(b) \geq l \ \min_k \ \operatorname{val} p_k(b) + 1$. □

Lemma 2.2. If $p = (p_1, \ldots, p_r)$ is a regular system of parameters of $\mathbb{C}[z_1, \ldots, z_r]$ and $\gamma \in \mathbb{A}^r_{\mathbb{C}}$, there exist $q \geq 1$ and $b \in \mathbb{A}^r_{\mathbb{C}[[T]]}$ such that $b(0) = 0$ and $p(b) = T^q \, \gamma$.

Proof. Let $K := \bigcup_{q \geq 1} \mathbb{C}((T^{\frac{1}{q}}))$ be the quotient field of the ring $\bigcup_{q \geq 1} \mathbb{C}[[T^{\frac{1}{q}}]]$ of all formal Puiseux series. By Newton-Puiseux theorem (see for example proposition 4.4 in [24]), K is an algebraic closure of $\mathbb{C}((T))$. Let us note that p is a regular system of parameters of $R := K[z_1, \ldots, z_r]$. By corollary 2.2, there exists $a \in \mathbb{A}^r_K$ such that $p(a) = T\gamma$. Let $q \geq 1$ be such that $b := a(T^q) \in \mathbb{A}^r_{\mathbb{C}((T))}$. Replacing T by T^q, we get $p(b) = T^q \gamma$. Using the valuative criterion of properness (see for example II, 4.7 in [12]), it is clear that $b \in \mathbb{A}^r_{\mathbb{C}[[T]]}$. Since $p(b(0)) = 0$, by (i) of the regularity criterion, we get $b(0) = 0$. □

3. Technical definitions.

If $m, n \geq 1$ are fixed, we set $N := (m + 1)(n + 1) - 1$.

Let $A_0, \ldots, A_m, B_1, \ldots, B_n$ be indeterminates and let $\mathbb{C}[B]$ (resp. $\mathbb{C}[A, B]$) be the polynomial algebra generated by the B_j (resp. by the A_i, B_j). We grade these polynomial algebras by assigning A_i, B_i to be homogeneous of degree i.

We will now successively define homogeneous polynomials:

- $C_i \in \mathbb{C}[A, B]$ of degree i for $1 \leq i \leq N$;
- $U_{i,j} \in \mathbb{C}[B]$ of degree $i - j$ for $1 \leq i \leq N$, $0 \leq j \leq m$;
- $W_{i,j} \in \mathbb{C}[B]$ of degree $i - j$ for $1 \leq i, j \leq m$;
- $D_i \in \mathbb{C}[B],\ E_i \in \mathbb{C}[A, B]$ of degree i for $1 \leq i \leq m$;
- $F_i \in \mathbb{C}[B],\ G_i \in \mathbb{C}[A, B]$ of degree i for $1 \leq i \leq N$;

satisfying the following points:

$$(0) \quad \sum_{1 \leq i \leq N} C_i X^{i+1} = A \circ B\,(X) - A_0 X$$

where $A(X) := \sum_{0 \leq i \leq m} A_i X^{i+1}$ and $B(X) := X + \sum_{1 \leq i \leq n} B_i X^{i+1}$;

(1) $A_i = A_0 D_i + E_i$ for $1 \leq i \leq m$;

(2) $C_i = A_0 F_i + G_i$ for $1 \leq i \leq N$;

(3) $E_i = \sum_{1 \leq j \leq m} W_{i,j}\, C_j$ for $1 \leq i \leq m$;

(4) $G_i := \sum_{1 \leq j \leq m} U_{i,j} E_j$ for $1 \leq i \leq N$;

(5) the matrices $(U_{i,j})_{1 \leq i, j \leq m}$ and $(W_{i,j})_{1 \leq i, j \leq m}$ are inverses of one another.

Later on, we will only use the above points (0-5) so that the reader in a hurry can skip the following technical construction.

Construction.

- The $C_i \in \mathbb{C}[A, B]$ are uniquely defined by (0).
- The $U_{i,j} \in \mathbb{C}[B]$ are uniquely defined by $C_i = \sum_{0 \leq j \leq m} U_{i,j} A_j$ $(1 \leq i \leq N)$.

Matricially, this may be written $\mathsf{C} = \mathsf{U}.\mathsf{A}$, where A (resp. C) is the column vector $\mathsf{A} = (A_i)_{0 \leq i \leq m}$ (resp. $\mathsf{C} = (C_i)_{1 \leq i \leq N}$) and U is the matrix $\mathsf{U} = (U_{i,j})_{\substack{1 \leq i \leq N \\ 0 \leq j \leq m}}$.

We have $\mathsf{U} = \begin{pmatrix} & \mathsf{V} \\ \mathsf{B} & \\ & * \end{pmatrix}$ where B is the column vector $\mathsf{B} = {}^t(B_1, \ldots, B_n, 0, \ldots, 0)$ and $\mathsf{V} := (U_{i,j})_{1 \leq i, j \leq m}$ is lower triangular with 1's on the diagonal.

- The matrix $\mathsf{W} = (W_{i,j})_{1 \leq i, j \leq m}$ is defined as the inverse of V.
- The column vectors $\mathsf{D} = (D_i)_{1 \leq i \leq m}$ and $\mathsf{E} = (E_i)_{1 \leq i \leq m}$ are defined by $\mathsf{D} := -\mathsf{W}.\widetilde{\mathsf{B}}$ and $\mathsf{E} := \mathsf{W}.\widetilde{\mathsf{C}}$, where $\widetilde{\mathsf{B}}$ (resp. $\widetilde{\mathsf{C}}$) denotes the column vector obtained from B (resp. C) by keeping its first m rows. Since the column vector $\widetilde{\mathsf{A}} := {}^t(A_1, \ldots, A_m)$ satisfies $\widetilde{\mathsf{C}} = A_0\widetilde{\mathsf{B}} + \mathsf{V}.\widetilde{\mathsf{A}}$, we get $\widetilde{\mathsf{A}} = -A_0\mathsf{W}.\widetilde{\mathsf{B}} + \mathsf{W}.\widetilde{\mathsf{C}}$, i.e. $\widetilde{\mathsf{A}} = A_0\mathsf{D} + \mathsf{E}$, which is (1).

- F_i, G_i are defined by $F_i := U_{i,0} + \sum\limits_{1 \le j \le m} U_{i,j} D_j$ and $G_i := \sum\limits_{1 \le j \le m} U_{i,j} E_j$ $(1 \le i \le N)$.

Since $C_i = \sum\limits_{0 \le j \le m} U_{i,j} A_j$ for $1 \le i \le N$ and $A_j = A_0 D_j + E_j$ for $1 \le j \le m$, we get $C_i = U_{i,0} A_0 + \sum\limits_{1 \le j \le m} U_{i,j}(A_0 D_j + E_j)$ and (2) follows. The assertions (3), (4) and (5) are obvious.

Remark. We have $G_i = \sum\limits_{1 \le j, k \le m} U_{i,j}\, W_{j,k}\, C_k$, so that if $i \le m$, we get $G_i = C_i$ and $F_i = 0$.

We will always use $R(m,n)$ through the next statement.

Lemma 2.3. If $R(m,n)$ is satisfied, the family $(F_{m+i})_{1 \le i \le n}$ is a regular system of parameters of $\mathbb{C}[B]$.

Proof. If $\overline{b} = (\overline{b}_1, \ldots, \overline{b}_n) \in \mathbb{C}^n$ satisfies $F_{m+i}(\overline{b}) = 0$ for $1 \le i \le n$, we want to show that $\overline{b} = 0$. Let us set $\overline{a}_0 = 1$, $\overline{a}_i = D_i(\overline{b})$ for $1 \le i \le m$ and $\overline{a} = (\overline{a}_0, \ldots, \overline{a}_m)$. By (1) evaluated at $(\overline{a}, \overline{b})$, we get $E_i(\overline{a}, \overline{b}) = 0$ for $1 \le i \le m$. By (3) and (5), we get $C_i(\overline{a}, \overline{b}) = 0$ for $1 \le i \le m$. Moreover, for any $m+1 \le i \le m+n$, we get $G_i(\overline{a}, \overline{b}) = 0$ by (4), so that $C_i(\overline{a}, \overline{b}) = 0$ by (2). For the moment, we have proved that $C_i(\overline{a}, \overline{b}) = 0$ for $1 \le i \le m+n$.

Let us set $a := X(1 + \overline{a}_1 X + \ldots + \overline{a}_m X^m)$, $b := X(1 + \overline{b}_1 X + \ldots + \overline{b}_n X^n)$ and $c := a \circ b = X(1 + \overline{c}_1 X + \ldots \overline{c}_N X^N) \in \mathbb{C}[X]$, where the $\overline{c}_i \in \mathbb{C}$. Using the relation (0), we get $\overline{c}_i = C_i(\overline{a}, \overline{b}) = 0$ for $1 \le i \le m+n$. By $R(m,n)$, we get $a = b = X$. □

We finish this section by a technical result to be used in lemma 3.1 below. Even if the proof relies on the above technical definitions, the statement does not.

Lemma 2.4. Let $a(X) = \sum\limits_{0 \le i \le m} a_i X^{i+1}$ and $b(X) = X + \sum\limits_{1 \le i \le n} b_i X^{i+1} \in \mathbb{C}((T))[X]$, where the $a_i, b_i \in \mathbb{C}((T))$. Let us set $c(X) := a \circ b\,(X) - a_0 X \in \mathbb{C}((T))[X]$. If $R(m,n)$ is satisfied, $\lim\limits_{T \to 0} b = X$ and $\lim\limits_{T \to 0} c = p \in \mathbb{C}[X]$, then $\deg p \le m+n+1$.

Proof. If $\mathsf{a} := (a_0, \ldots, a_m)$ and $\mathsf{b} := (b_1, \ldots, b_n)$, we have $c(X) = \sum\limits_{1 \le i \le N} C_i(\mathsf{a}, \mathsf{b})\, X^{i+1}$ by the relation (0). Since $\lim\limits_{T \to 0} b = X$ all the b_i belong to $T\mathbb{C}[[T]]$ and in the same way, all the $C_i(\mathsf{a}, \mathsf{b})$ belong to $\mathbb{C}[[T]]$.

<u>Claim.</u> If $m+1 \le i \le N$, then $G_i(\mathsf{a}, \mathsf{b}) \in T\mathbb{C}[[T]]$.

This comes from the relation $G_i = \sum\limits_{1 \le j, k \le m} U_{i,j}\, W_{j,k}\, C_k$ because, if $1 \le j,\, k \le m$:

- $U_{i,j}(\mathsf{b}) \in T\mathbb{C}[[T]]$ since $U_{i,j} \in \mathbb{C}[B]$ is homogeneous of degree $i - j \ge m+1-m = 1$;

• $W_{j,k}(\mathsf{b}) \in \mathbb{C}[[T]]$ since $W_{j,k}$ is a polynomial;
• $C_k(\mathsf{a},\mathsf{b}) \in \mathbb{C}[[T]]$.

By (2), we have $C_i(\mathsf{a},\mathsf{b}) = a_0 F_i(\mathsf{b}) + G_i(\mathsf{a},\mathsf{b})$.

If $i \geq m+1$, we have $\operatorname{val} C_i(\mathsf{a},\mathsf{b}) \geq 0$ and $\operatorname{val} G_i(\mathsf{a},\mathsf{b}) \geq 1$, so that $\operatorname{val} a_0 F_i(\mathsf{b}) \geq 0$. We want to show that $\operatorname{val} C_i(\mathsf{a},\mathsf{b}) \geq 1$, when $i > m+n$. For this, it is sufficient to show that $\operatorname{val} a_0 F_i(\mathsf{b}) \geq 1$.

By lemmas 2.3 and 2.1, we have $\operatorname{val} F_i(\mathsf{b}) \geq \min_{1\leq j\leq n} \operatorname{val} F_{m+j}(\mathsf{b}) + 1$ (for $i > m+n$) so that $\operatorname{val} a_0 F_i(\mathsf{b}) \geq \min_{1\leq j\leq n} \operatorname{val} a_0 F_{m+j}(\mathsf{b}) + 1 \geq 1$. □

III. PROOF OF THEOREM B.

In all this section, $m, n \geq 1$ are fixed integers and we assume that $R(m,n)$ is satisfied. If we set $d = (d_1, d_2) = (m+1, n+1)$, we want to show that $\overline{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$.

1. The first inclusion.

If $f \in \overline{\mathcal{G}}_d$, let us show that $f \in \bigcup_{e \preceq d} \mathcal{G}_e$. By [7], the length is a lower semicontinuous function on $\mathcal{G}$ so that the length of f satisfies $l \leq 2$. We will consider 3 cases:
• $l = 0$. There is nothing to show;
• $l = 1$. We conclude by lemma 3.1 below;
• $l = 2$. We conclude by lemma 3.2 below.

Lemma 3.1. If $e \geq 2$ and $\mathcal{G}_{(e)} \cap \overline{\mathcal{G}}_d \neq \emptyset$, then $e < d_1 + d_2$.

Proof. If $f \in \mathcal{G}_{(e)} \cap \overline{\mathcal{G}}_d$, let us prove that $e < d_1 + d_2$. Since $\mathcal{A}.f.\mathcal{A} \subseteq \overline{\mathcal{G}_d}$, we may assume that $f = (X + p(Y), Y)$ with $\deg p = e$. If $e \leq d_2$, there is nothing to prove. So, let us assume that $e > d_2$.

By corollary 2.1, there exists $g = (g_1, g_2) \in \mathcal{G}_d\Big(\mathbb{C}((T))\Big)$ such that $f = \lim_{T\to 0} g_T$.

We must have $\deg g_1 = d_1 d_2$. Let us show that we may assume that $\deg g_2 = d_2$. Indeed, there exists a unique $\lambda \in \mathbb{C}((T))$ such that $\deg(g_2 - \lambda g_1) = d_2$ (since g is of multidegree d). It is enough to show that $\operatorname{val}(\lambda) > 0$, because we can then replace g by $(u_1, u_2) := (g_1, g_2 - \lambda g_1)$. Let μ (resp. ν) $\in \mathbb{C}((T))$ be the Y^e-coefficient of g_2 (resp. u_1). Applying the equality $g_2 = u_2 + \lambda u_1$ to the Y^e-coefficient, we get $\mu = \lambda\nu$ (since we have assumed that $e > d_2$). However, we have $\operatorname{val}(\nu) = 0$ (since $\lim_{T\to 0} u_1(T) = X + p(Y)$ and $\deg p = e$), so that $\operatorname{val}(\lambda) = \operatorname{val}(\mu) > 0$.

Since $\deg g_1 > \deg g_2$, it is well-known (see for example theorem 1 (i) in [6]) that we can write (in a unique way)

$$g = \tau.t_1.\sigma.t_2.l,$$

where $\tau = (X + a, Y + b)$ is a translation,

$t_1 = \left(X + \sum_{0 \le i \le m} a_i Y^{i+1}, Y \right)$, $t_2 = \left(X + \sum_{1 \le i \le n} b_i Y^{i+1}, Y \right)$ are triangular automorphisms,

$\sigma = (Y, X)$ and $l = (l_1, l_2) = (\alpha X + \beta Y, \gamma X + \delta Y)$ are linear automorphisms, with the a_i's, b_i's, $a, b, \alpha, \beta, \gamma, \delta$ belonging to $\mathbb{C}((T))$.

By doing the composition, this can be written:

$$g = \left(l_2 + \sum_{0 \le i \le m} a_i \left[l_1 + \sum_{1 \le j \le n} b_j \, l_2^{j+1} \right]^{i+1} + a, \quad l_1 + \sum_{1 \le j \le n} b_j \, l_2^{j+1} + b \right).$$

Since $f = \lim_{T \to 0} g_T$, we get $a, b \in \mathbb{C}[[T]]$ and there is no restriction to assume that $a = b = 0$. Let us show that we may assume that $l = (l_1, l_2) = (Y, X + \rho Y)$ for some $\rho \in \mathbb{C}((T))$. First of all, let us note that $\lim_{T \to 0} \alpha = 0$, $\lim_{T \to 0} \beta = 1$ and $\lim_{T \to 0} \alpha\delta - \beta\gamma = -1$. The last relation comes from the Jacobian equality $\operatorname{Jac} g = \operatorname{Jac} \sigma \times \operatorname{Jac} l = -(\alpha\delta - \beta\gamma)$.

Let us set $\rho := \dfrac{\delta - \alpha}{\beta}$. Since $(l_1, l_2) = (Y, X + \rho Y).(-\rho \, l_1 + l_2, l_1)$, it is enough to show that $\lim_{T \to 0} h_T = (X, Y)$, where $h_T := (-\rho \, l_1 + l_2, l_1)$. For the second component, it is clear. For the first, we have $-\rho \, l_1 + l_2 = (\gamma - \rho\alpha)X + (\delta - \rho\beta)Y$. But $\gamma - \rho\alpha = \dfrac{\alpha^2}{\beta} - \dfrac{\alpha\delta - \beta\gamma}{\beta}$, so that $\lim_{T \to 0} \gamma - \rho\alpha = 1$ and $\delta - \rho\beta = \alpha$, so that $\lim_{T \to 0} \delta - \rho\beta = 0$.

So, we can now assume that:

$$g = \left(X + \rho Y + \sum_{0 \le i \le m} a_i \left[Y + \sum_{1 \le j \le n} b_j \, (X + \rho Y)^{j+1} \right]^{i+1}, \quad Y + \sum_{1 \le j \le n} b_j \, (X + \rho Y)^{j+1} \right).$$

Inspecting the Y-powers, the relation $\lim_{T \to 0} g_T = (X + p(Y), Y)$ gives us:

$$\lim_{T \to 0} \rho Y + \sum_{0 \le i \le m} a_i \left[Y + \sum_{1 \le j \le n} b_j \, \rho^{j+1} Y^{j+1} \right]^{i+1} = p(Y) \quad \text{and} \quad \lim_{T \to 0} \sum_{1 \le j \le n} b_j \, \rho^{j+1} Y^{j+1} = 0.$$

Setting $\widetilde{b}_j := b_j \rho^{j+1}$, we get:

$$\lim_{T \to 0} \rho Y + \sum_{0 \le i \le m} a_i \left[Y + \sum_{1 \le j \le n} \widetilde{b}_j \, Y^{j+1} \right]^{i+1} = p(Y) \quad \text{and} \quad \lim_{T \to 0} \sum_{1 \le j \le n} \widetilde{b}_j Y^{j+1} = 0.$$

Looking at the Y-coefficient, the first relation shows us that $\lim_{T \to 0} \rho + a_0 = p_1$, where p_1 is the Y-coefficient of $p(Y)$.

Therefore $\lim_{T \to 0} \sum_{0 \le i \le m} a_i \left[Y + \sum_{1 \le j \le n} \widetilde{b}_j \, Y^{j+1} \right]^{i+1} - a_0 Y = p(Y) - p_1 Y$ and lemma 2.4 tells us that $\deg \Big(p(Y) - p_1 Y \Big) \le m + n + 1 = d_1 + d_2 - 1$. $\square$

Lemma 3.2. If $\mathcal{G}_{(e_1,e_2)} \cap \overline{\mathcal{G}}_{(d_1,d_2)} \neq \emptyset$, then $e_1 \leq d_1$ and $e_2 \leq d_2$.

Proof. Let us denote by $V = \underset{j\geq 0}{\oplus} V_j$ the polynomial ring $\mathbb{C}[X,Y]$ graded by degree. For each $k \geq 0$, let $\Pi_{>k} : V \to V$ be the projection along $\underset{j\leq k}{\oplus} V_j$ onto $\underset{j>k}{\oplus} V_j$.

We recall that $u, v \in V$ are linearly dependant if and only if $u \wedge v = 0$ (in $\bigwedge^2 V$).

The key point is the fact that for each $f = (f_1, f_2) \in \mathcal{G}_{(d_1,d_2)}$, we have

$$d_2 = \min\{k,\ \Pi_{>k}(f_1) \wedge \Pi_{>k}(f_2) = 0\}.$$

Therefore, if $f \in \mathcal{G}_{(e_1,e_2)} \cap \overline{\mathcal{G}}_{(d_1,d_2)}$, we have $e_2 \leq d_2$ (since $\Pi_{>d_2}(f_1) \wedge \Pi_{>d_2}(f_2) = 0$). The map $g \mapsto g^{-1}$ being an automorphism of (the infinite dimensional algebraic variety) $\mathcal{G}$, we also have $f^{-1} \in \mathcal{G}_{(e_2,e_1)} \cap \overline{\mathcal{G}}_{(d_2,d_1)}$ so that $e_1 \leq d_1$. □

2. The second inclusion.

Let us show that $\mathcal{G}_e \subseteq \overline{\mathcal{G}}_d$ for any $e \preceq d$. If the length of e is 0 or 2, it is obvious. Therefore, the conlusion follows from the next result:

Lemma 3.3. If $2 \leq e < d_1 + d_2$, then $\mathcal{G}_{(e)} \subseteq \overline{\mathcal{G}}_d$.

Proof. It is enough to show that $\left(X + \sum_{1\leq i\leq m+n} \gamma_i Y^{i+1},\quad Y \right) \in \overline{\mathcal{G}}_d$ for any $\gamma_i \in \mathbb{C}$.

We take back the notations of §2.3. By lemmas 2.3 and 2.2, there exist $q \geq 1$ and $\widetilde{b} := (\widetilde{b}_1, \ldots, \widetilde{b}_n) \in \mathbb{A}^n_{\mathbb{C}[[T]]}$ such that $\widetilde{b}(0) = 0$ and $\dfrac{F_{m+i}(\widetilde{b})}{T^q} = \gamma_{m+i}$ (for $1 \leq i \leq n$). Let us now set $a_0 := \dfrac{1}{T^q}$, $a_i := a_0 D_i(\widetilde{b})$ for $1 \leq i \leq m$, $\rho := -\dfrac{1}{T^q}$ and $b_j := \dfrac{\widetilde{b}_j}{\rho^{j+1}}$ for $1 \leq j \leq n$.

If we set $g := t_1.\sigma.t_2.l$, where $t_1 = \left(X + \sum_{0\leq i\leq m} a_i Y^{i+1}, Y \right)$, $t_2 = \left(X + \sum_{1\leq i\leq n} b_i Y^{i+1}, Y \right)$ are triangular automorphisms, $\sigma = (Y, X)$ and $l = (Y, X + \rho Y)$, then:

$$g = \left(X + \rho Y + \sum_{0\leq i\leq m} a_i \left[Y + \sum_{1\leq j\leq n} b_j\,(X+\rho Y)^{j+1} \right]^{i+1},\quad Y + \sum_{1\leq j\leq n} b_j\,(X+\rho Y)^{j+1} \right).$$

It is now easy (and technical) to check that $\lim_{T\to 0} g_T = \left(X + \sum_{m+1\leq i\leq m+n} \gamma_i Y^{i+1},\quad Y \right)$.

Let us begin to show that $\lim_{T\to 0} g_2 = Y$.

For $1 \leq j \leq n$, we have $b_j (X+\rho Y)^{j+1} = \widetilde{b}_j (\frac{1}{\rho} X + Y)^{j+1}$, where $\lim_{T\to 0} \widetilde{b}_j = \lim_{T\to 0} \frac{1}{\rho} = 0$, so that $\lim_{T\to 0} b_j (X+\rho Y)^{j+1} = 0$ and the result is clear.

Let us now deal with $g_1 = X + \rho Y + \sum_{0 \le i \le m} a_i\, g_2^{i+1}$.

<u>First step.</u> Let us show that in this last expression of g_1, we can replace g_2 by

$$p := Y + \sum_{1 \le j \le n} \widetilde{b}_j Y^{j+1}.$$

It is sufficient to check that $\lim_{T\to 0} a_i(g_2^{i+1} - p^{i+1}) = 0$.

As $\lim_{T\to 0} g_2 = \lim_{T\to 0} p = Y$, we will only check that $\lim_{T\to 0} a_i\,(g_2 - p) = 0$.

Since $g_2 - p = \sum_{1 \le j \le n} b_j \left[(X + \rho Y)^{j+1} - (\rho Y)^{j+1}\right]$, it is enough to show that:

$$\lim_{T\to 0} a_i b_j \left[(X + \rho Y)^{j+1} - (\rho Y)^{j+1}\right] = 0.$$

As $\lim_{T\to 0} \dfrac{(X + \rho Y)^{j+1} - (\rho Y)^{j+1}}{(j+1) X Y^j \rho^j} = 1$, we will only show that $\lim_{T\to 0} a_i b_j \rho^j = 0$.

It is clear, because $a_i b_j \rho^j = -\dfrac{a_i}{a_0}\widetilde{b}_j$ where $\lim_{T\to 0} \widetilde{b}_j = 0$ and $\lim_{T\to 0} \dfrac{a_i}{a_0} = 1$ (resp. 0) if $i = 0$ (resp. $i \ge 1$).

<u>Second step.</u> Let us show that $\lim_{T\to 0} c = \sum_{m+1 \le i \le m+n} \gamma_i\, Y^{i+1}$, where

$$c := \sum_{0 \le i \le m} a_i\, p^{i+1} - a_0 Y.$$

If $\mathsf{a} := (a_0, \ldots, a_m)$, by the relation (0) we have $c = \sum_{1 \le i \le N} C_i(\mathsf{a}, \widetilde{b}) Y^{i+1}$.

We get $E_j(\mathsf{a}, \widetilde{b}) = 0$ for $1 \le j \le m$ by (1), so that $G_i(\mathsf{a}, \widetilde{b}) = 0$ for $1 \le i \le N$ by (4) and $C_i(\mathsf{a}, \widetilde{b}) = a_0 F_i(\widetilde{b}) = \dfrac{F_i(\widetilde{b})}{T^q}$ for $1 \le i \le N$ by (2). Therefore:

- $C_i(\mathsf{a}, \widetilde{b}) = 0$ for $1 \le i \le m$;
- $\lim_{T\to 0} C_{m+i}(\mathsf{a}, \widetilde{b}) = \lim_{T\to 0} \dfrac{F_{m+i}(\widetilde{b})}{T^q} = \gamma_{m+i}$ for $1 \le i \le n$;
- $\lim_{T\to 0} C_i(\mathsf{a}, \widetilde{b}) = 0$ for $i > m+n$, since $\operatorname{val} F_i(\widetilde{b}) \ge \min_{1 \le j \le n} \operatorname{val} F_{m+j}(\widetilde{b}) + 1 \ge q + 1$ (by lemma 2.1).

If we now set $f := t.g$, where $t := \left(X + \sum_{1 \le i \le m} \gamma_i Y^{i+1}, Y\right)$ is a triangular automorphism, then $f \in \mathcal{G}_d\left(\mathbb{C}((T))\right)$ and $\lim_{T\to 0} f_T = \left(X + \sum_{1 \le i \le m+n} \gamma_i Y^{i+1}, \quad Y\right)$. □

IV. PROOF OF THEOREM D.

If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ as in theorem D, let us show that $\mathcal{H}=\mathcal{G}$. It is easy to see that $\mathcal{H}$ contains a triangular automorphism $f=(X+p(Y),Y)$ with $\deg p\geq 2$. If we set $g_\alpha:=(X,Y+\alpha)\in\mathcal{H}$ ($\alpha\in\mathbb{C}$), then the commutator $[f,g_\alpha]:=f.g_\alpha.f^{-1}.g_\alpha^{-1}\in\mathcal{H}$ is equal to $(X+q(Y),Y)$, where $q(Y):=p(Y)-p(Y-\alpha)$. If α is well chosen, one may assume that $\deg q=\deg p-1$. Therefore, by a decreasing induction, we see that $\mathcal{H}$ contains a triangular automorphism of degree 2.

Since $\mathcal{G}_{(2)}=\mathcal{A}.(X+Y^2,Y).\mathcal{A}$ (exercise), we get $\mathcal{G}_{(2)}\subseteq\mathcal{H}$.

By another induction, we get $\mathcal{G}_{(d)}\subseteq\mathcal{H}$ for any $d\geq 2$. Indeed, if $\mathcal{G}_{(d)}\subseteq\mathcal{H}$, we get $\mathcal{G}_{(d,2)}\subseteq\mathcal{H}$, so that $\overline{\mathcal{G}}_{(d,2)}\subseteq\overline{\mathcal{H}}=\mathcal{H}$. But $\mathcal{G}_{(d+1)}\subseteq\overline{\mathcal{G}}_{(d,2)}$ by theorem C.

It is now clear that $\mathcal{H}=\mathcal{G}$. $\square$

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References

[1] S.S. Abhyankar, T.T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math., 276 (1975), 148-166.

[2] H. Bass, E. Connell, D. Wright, The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse, Bull. of the A.M.S., 7 (1982), 287-330.

[3] E. Edo, J.-P. Furter, Some families of polynomial automorphisms, J. of Pure and Applied Algebra 194 (2004), 263-271.

[4] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995.

[5] S. Friedland, J. Milnor, Dynamical properties of plane polynomial automorphisms, Ergod. Th & Dyn. Syst. 9 (1989), 67-99.

[6] J.-P. Furter, On the variety of automorphisms of the affine plane, J. Algebra 195 (1997), 604-623.

[7] J.-P. Furter, On the length of automorphisms of the affine plane, Math. Ann. 322 (2002), 401-411.

[8] J.-P. Furter, Jet Groups, J. Algebra 315 (2007), no. 2, 720-737.

[9] J.-P. Furter, Plane polynomial automorphisms of fixed multidegree, Math. Ann. 343 (2009), 901-920.

[10] M. H. Gizatullin, V. I. Danilov, Automorphisms of affine surfaces I (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 3, 523-565, 703.

[11] A. Grothendieck, J. Dieudonné, Eléments de géométrie algébrique, II Etude globale élémentaire de quelques classes de morphismes, Publ. Math. IHES 8 (1961).

[12] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[13] A. Hurwitz, Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche, Math. Ann. 46 (1891), 279-284.

[14] H. W. E. Jung, Über ganze birationale Transformationen der Ebene, J. Reine Angew. Math. 184 (1942), 161-174.

[15] T. Kambayashi, Automorphism group of a polynomial ring and algebraic group action on an affine space, J. Algebra 60 (1979), no. 2, 439-451.

[16] W. van der Kulk, On polynomial rings in two variables, Nieuw. Arch. Wisk. (3) 1 (1953), 33-41.

[17] J. Liouville, Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationnelles algébriques, C.R. Acad. Sci. Paris, 18 (1844), 883-885.

[18] J. Liouville, Nouvelle démonstration d'un théorème sur les irrationnelles algébriques inséré dans le compte rendu de la dernière séance, C.R. Acad. Sci. Paris, 18 (1844), 910-911.

[19] S. Mallat, A wavelet tour of signal processing, 2nd edition. Academic Press, Inc., San Diego, CA, 1999.

[20] D. Mumford, J. Fogarty, F. Kirwan, Geometric invariant theory. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], 34. Springer-Verlag, Berlin, 1994.

[21] M. Nagata, On automorphism group of k[x,y], Lecture Notes in Math., Kyoto Univ., 5, (1972).

[22] J. F. Ritt, Prime and composite polynomials. Trans. Amer. Math. Soc. 23 (1922), no. 1, 51-66.

[23] K. F. Roth, Rational Approximations to Algebraic Numbers, Mathematika 2 (1955), 1-20; corrigendum, 168.

[24] J. M. Ruiz, The basic theory of power series. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1993.

[25] I. R. Shafarevich, On some infinite-dimensional groups, Rend. Mat. e Appl. (5) 25 (1966), no. 1-2, 208-212.

[26] I. R. Shafarevich, On some infinite-dimensional groups II, Math. USSR Izv., 18 (1982), 214-226.

[27] I. R. Shafarevich, Letter to the editors: "On some infinite-dimensional groups. II", (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 59 (1995), no. 3, 224.

[28] M. Suzuki, Propriétés topologiques des polynômes de deux variables complexes et automorphismes algébriques de l'espace C^2, J. Math. Soc. Japan. 26 (1974), 241-257.

[29] H. Weyl, The Theory of Groups and Quantum Mechanics. Dutton, New York, 1931.

[30] D. Wright, Abelian subgroups of Aut_k(k[X,Y]) and applications to actions on the affine plane, Illinois J. Math., 23.4 (1979), 579-634.