Fat Point Embeddings in Affine Space.

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Abstract.

We show that any fat point (local punctual scheme) has at most one embedding in the affine space up to analytic equivalence. If the algebra of functions of the fat point admits a non-trivial grading over the nonnegative integers, we prove that it has at most one embedding up to algebraic equivalence. However, we give an example of a fat point having algebraically non equivalent embeddings in the affine plane.

Keywords.

Affine space, Automorphisms, Embeddings, Fat point.

Mathematics Subject Classification (2000).

14R10.

INTRODUCTION.

Let $\mathbb{A}^N$ be the complex affine space of dimension N. It is well known that all algebraic embeddings of $\mathbb{A}^1$ in $\mathbb{A}^2$ are equivalent (see [1]). Yet, it is proven in [5] that it is no longer true for analytic embeddings. In this paper, we consider fat points embeddings in $\mathbb{A}^N$. By a fat point, we mean any complex affine scheme whose algebra of functions is finite and local. Let us recall that an analytic embedding of a fat point in $\mathbb{A}^N$ is in fact algebraic. A $\mathbb{Z}$-grading $B = \bigoplus_k B_k$ of an algebra B is called non-trivial if $B_0 \neq B$. Here are our three main results:

Theorem A. Up to analytic equivalence, a fat point has at most one embedding in $\mathbb{A}^N$.

Theorem B. Up to algebraic equivalence, a fat point whose algebra of functions admits a non-trivial $\mathbb{Z}_+$-grading has at most one embedding in $\mathbb{A}^N$.

Theorem C. There exists a fat point admitting algebraically non equivalent embeddings in $\mathbb{A}^2$.

I. NOTATIONS AND PRELIMINARY RESULTS.

1. Mono-embeddability.

Definition. 1. Let us recall that two morphisms f_1 and $f_2 : X \to Y$ in a category $\mathcal{C}$ are said to be equivalent if there exist automorphisms α (resp. β) of X (resp. Y) such that the following diagramm is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{f_2} & Y \end{array}$$

2. A scheme X is said to be algebraically mono-embeddable in a scheme Y if, in the category of algebraic schemes, all (closed) embeddings of X in Y are equivalent. This is equivalent to saying that if X_1 and X_2 are closed subschemes of Y which are isomorphic to X, then there exists an automorphism β of Y such that $\beta(X_1) = X_2$.

3. A scheme X is said to be algebraically strongly mono-embeddable in a scheme Y if, in the category of algebraic schemes, for all (closed) embeddings $f_1, f_2 : X \to Y$ there exists an automorphism β of Y such that $f_2 = \beta \circ f_1$. This is equivalent to saying that if X_1 and X_2 are closed subschemes of Y which are isomorphic to X, then any isomorphism $f : X_1 \to X_2$ can be extended in an automorphism β of Y.

Remarks. 1. Strong mono-embeddability implies mono-embeddability.
2. Definitions 2 and 3 could also have been set in the analytic case.

Examples. 1. If $N = 2$ or $N \geq 4$, then $\mathbb{A}^1$ is strongly algebraically mono-embeddable in $\mathbb{A}^N$ (see [1] or [10]). The case $N = 3$ is still unsolved. Furthermore, $\mathbb{A}^1$ is not analytically mono-embeddable in $\mathbb{A}^2$ (see [5]).

2. Other examples of curves which are (resp. which are not) algebraically mono-embeddable in $\mathbb{A}^2$ are given in [2] (resp. in [3] and [12]).

3. Let us recall that any smooth affine (not necessarily irreducible) variety of dimension d can be embedded as a closed subscheme in $\mathbb{A}^{2d+1}$. If $N > 2d + 1$, these varieties are strongly algebraically mono-embeddable in $\mathbb{A}^N$ by [14].

2. Jets.

Let $\widehat{E}$ be the monoid of analytic endomorphisms of the analytic germ $(\mathbb{A}^N, 0)$ and let E (resp. $\widetilde{E}$) be the submonoid of algebraic (resp. analytic) endomorphisms of the pointed variety $(\mathbb{A}^N, 0)$. Let $\widehat{R} := \mathbb{C}\{x_1, \ldots, x_N\}$ be the algebra of convergent power series in the indeterminates $x_1, \ldots, x_N$ and let R (resp. $\widetilde{R}$) be the subalgebra of polynomial (resp. analytic) functions on $\mathbb{A}^N$. We have $R = \mathbb{C}[x_1, \ldots, x_N]$. We will identify any element f of $\widehat{E}$ (resp. E, resp. $\widetilde{E}$) with its coordinate functions $f = (f_1, \ldots, f_N)$ where each $f_L \in \widehat{R}$ (resp. R, resp. $\widetilde{R}$) satisfies $f_L(0) = 0$. If F is a monoid, let us denote by F^* its group of invertible elements. Let $\widehat{A} := \widehat{E}^*$ be the group of analytic automorphisms of the analytic germ $(\mathbb{A}^N, 0)$ and let $A := E^*$ (resp. $\widetilde{A} := \widetilde{E}^*$) be the group of algebraic (resp. analytic) automorphisms of the pointed variety $(\mathbb{A}^N, 0)$. We have: $R \subset \widetilde{R} \subset \widehat{R}$, $E \subset \widetilde{E} \subset \widehat{E}$ and $A \leq \widetilde{A} \leq \widehat{A}$, where the relation $G_1 \leq G_2$ means that G_1 is a subgroup of G_2. If $f \in \widehat{E}$ and $n \geq 1$, let $J_n f := \sum_{0 \leq k \leq n} \frac{1}{k!} D^k f.x^k$ denote its n-jet at the origin, where $D^k f$ denotes the k-th differential of f at the origin, $x = (x_1, \ldots, x_N)$ and $x^k = (\underbrace{x, \ldots, x}_{k})$. If $J_n(E)$ (resp. $J_n(\widetilde{E})$, resp. $J_n(\widehat{E})$) is the space of all $J_n f$ when f describes E (resp. $\widetilde{E}$, resp. $\widehat{E}$), then $J_n(E) = J_n(\widetilde{E}) = J_n(\widehat{E})$ is naturally a monoid and $J_n(E) \simeq \bigoplus_{1 \leq k \leq n} E_k$, where E_k is the subspace of k-homogeneous elements of E. The map $J_n : \widehat{E} \to J_n(\widehat{E}) = J_n(E)$ being a monoid morphism, we get: $J_n(A) \leq J_n(\widetilde{A}) \leq J_n(\widehat{A}) \leq J_n(E)^*$.

Of course, $J_n r$ will denote the n-jet at the origin of $r \in R$. The space of all $J_n r$ when r describes R will be denoted by $J_n(R)$. We have $J_n(R) \simeq \bigoplus_{0 \leq k \leq n} R_k$, where R_k is the subspace of k-homogeneous elements of R. The Jacobian map $\mathrm{Jac} : E \to R$ induces a map $J_n(E) \to J_{n-1}(R)$. If f belongs to a graded object, let $f_{(k)}$ be its k-homogeneous component. Let GL be the linear group of $\mathbb{C}^N$. By [6], we get $J_n(A) = \{j \in J_n(E), \mathrm{Jac}\, j \in \mathbb{C}^*\}$ and $J_n(\widetilde{A}) = J_n(\widehat{A}) = J_n(E)^* = \{f \in J_n(E), J_1 f \in GL\}$.

3. A nice algebra.

Let us set $S_{n,N} := \mathbb{C}[x_1, \ldots, x_N]/(x_1, \ldots, x_N)^{n+1}$, where $n, N \geq 1$. If the dimension N is understood, we will denote this last algebra by S_n. Let us recall a basic property of commutative algebra (to be also used in the proof of lemma 2.1 below). Let T be the functor going from the category of finite local complex algebras to the category of finite dimensional complex vector spaces, associating to the local algebra $(B, \mathcal{N})$ the vector space $T(B) := \mathcal{N}/\mathcal{N}^2$. If $u \in \mathcal{N}$, let $\overline{u} \in T(B)$ be the class of u modulo $\mathcal{N}^2$. If $u_1, \ldots, u_m \in \mathcal{N}$, it is well known that the following assertions are equivalent:

(i) the ideal $\mathcal{N}$ is generated by $u_1, \ldots, u_m$;
(ii) the algebra B is generated by $u_1, \ldots, u_m$;
(iii) the vector space $T(B)$ is generated by $\overline{u}_1, \ldots, \overline{u}_m$.

Therefore, it is clear that the embedding dimension of B (i.e. the minimal number of generators of the algebra B) satisfies $ed(B) = \dim T(B)$.

If $\mathcal{M}$ is the maximal ideal of S_n, let $u_1, \ldots, u_N$ and $v_1, \ldots, v_N \in \mathcal{M}$. If $\overline{u}_1, \ldots, \overline{u}_N$ is a basis of $T(S_n)$, there exists a unique algebra morphism $f : S_n \to S_n$ such that $f(u_k) = v_k$ for each k. Furthermore, the three following assertions are equivalent:

(i) f is an algebra automorphism;
(ii) $T(f) : T(S_n) \to T(S_n)$ is a linear automorphism;
(iii) $\overline{v}_1, \ldots, \overline{v}_N$ is a basis of $T(S_n)$.

This proves that $Aut(S_n)$ is naturally isomorphic to $J_n(\widetilde{A})$. Let us note that any finite local complex algebra is the quotient of some $S_{n,N}$. Since the automorphism group of this last algebra is well understood, it seems attractive to study any quotient S_n/I via the nice S_n. The lifting lemma of next section will allow us to proceed in such a way.

Let us finish this subsection by computing the unipotent radical of $J_n(\widetilde{A})$. If G is a linear algebraic group, we recall that its unipotent radical $R_u(G)$ is by definition the largest connected normal unipotent subgroup of G (see for example § 19.5 of [9]).

Lemma 1.1. We have $R_u\left(J_n(\widetilde{A})\right) = \{f \in J_n(\widetilde{A}),\ J_1 f = \mathrm{id}\}$.

Proof. Let us set $G := J_n(\widetilde{A})$ and $H := \mathrm{Ker}(\varphi)$, where φ is the surjective morphism $\varphi:\ G \to GL,\ j \mapsto J_1(j)$. Since $G/H \simeq GL$ is reductive, we get $R_u(G) \leq H$.

Conversely, it is clear that H is unipotent. Indeed, G is naturally a closed subgroup of the linear group $GL(S_n)$ of the vector space S_n. Let $M := \{x^\alpha,\ \alpha \in \mathbb{N}^N,\ |\alpha| \leq n\}$ be the set of all monomials in $x_1, \ldots, x_N$ of degree less than or equal to n. Let us endow M with any order $\prec$ satisfying $|\alpha| < |\beta| \Longrightarrow x^\alpha \prec x^\beta$. If $f \in H$ and $x^\alpha \in M$, we have $f(x^\alpha) - x^\alpha \in \mathrm{Span}\left(x^\beta\right)_{x^\alpha \prec x^\beta}$. Therefore, the matrix of f in the basis x^α of S_n, where the x^α are taken with the order $\prec$, is lower triangular with ones on the diagonal. □

II. LIFTING LEMMA AND CONSEQUENCES.

Lemma 2.1 (lifting lemma). If I, J are ideals of S_n, then any algebra isomorphism $f:\ S_n/I \to S_n/J$ can be lifted to an algebra automorphism $\widehat{f}:\ S_n \to S_n$ satisfying $\widehat{f}(I) = J$. If π_I denotes the canonical surjection from S_n to S_n/I, this means that the following diagram is commutative:

$$\begin{array}{ccc} S_n & \xrightarrow{\widehat{f}} & S_n \\ \Big\downarrow{\scriptstyle \pi_I} & & \Big\downarrow{\scriptstyle \pi_J} \\ S_n/I & \xrightarrow{f} & S_n/J \end{array}$$

Proof. We may of course assume that I and J are different from S_n, so that I, J are included in $\mathcal{M}$. Since $S_n/I \simeq S_n/J$, we have $\mathrm{ed}(S_n/I) = \mathrm{ed}(S_n/J)$. But $\mathrm{ed}(S_n/I) = \dim T(S_n/I)$ and since $\pi_I:\ S_n \to S_n/I$ is onto, the maximal ideal of S_n/I is equal to $\pi_I(\mathcal{M})$. We have therefore

$$T(S_n/I) = \frac{\pi_I(\mathcal{M})}{\pi_I(\mathcal{M})^2} = \frac{\pi_I(\mathcal{M})}{\pi_I(\mathcal{M}^2)} \simeq \frac{\pi_I^{-1}\left(\pi_I(\mathcal{M})\right)}{\pi_I^{-1}\left(\pi_I(\mathcal{M}^2)\right)} = \frac{\mathcal{M}+I}{\mathcal{M}^2+I} = \frac{\mathcal{M}}{\mathcal{M}^2+I}.$$

But $\mathcal{M}^2 \subset \mathcal{M}^2 + I \subset \mathcal{M}$, so that

$\mathrm{ed}(S_n/I) = \dim \mathcal{M}/(\mathcal{M}^2+I) = \dim \mathcal{M}/\mathcal{M}^2 - \dim\,(\mathcal{M}^2+I)/\mathcal{M}^2$.

By the same way $\mathrm{ed}(S_n/J) = \dim \mathcal{M}/\mathcal{M}^2 - \dim\,(\mathcal{M}^2+J)/\mathcal{M}^2$, so that we can set $r := \dim\,(\mathcal{M}^2+I)/\mathcal{M}^2 = \dim\,(\mathcal{M}^2+J)/\mathcal{M}^2$.

Thanks to the natural isomorphism $(\mathcal{M}^2+I)/\mathcal{M}^2 \simeq I/(\mathcal{M}^2 \cap I)$, there exit $u_1, \ldots, u_r \in I$ such that $\overline{u}_1, \ldots, \overline{u}_r$ is a basis of $(\mathcal{M}^2+I)/\mathcal{M}^2$. By the same way, there exist $v_1, \ldots, v_r \in$

J such that $\overline{v}_1, \ldots, \overline{v}_r$ is a basis of $(\mathcal{M}^2 + J)/\mathcal{M}^2$. Let us choose $u_{r+1}, \ldots, u_N \in \mathcal{M}$ such that $\overline{u_1}, \ldots, \overline{u_N}$ is a basis of $\mathcal{M}/\mathcal{M}^2$. For each $k \geq r+1$, let us choose $v_k \in \pi_J^{-1}(f(\pi_I(u_k)))$. Let $\widehat{f} : S_n \to S_n$ be the algebra morphism defined by $\widehat{f}(u_k) = v_k$ for each k. By construction, we have $\pi_J(v_k) = f(\pi_I(u_k))$, i.e. $\pi_J\left(\widehat{f}(u_k)\right) = f(\pi_I(u_k))$, so that $\pi_J \circ \widehat{f} = f \circ \pi_I$. Let us now check that $\widehat{f}$ is an automorphism. We have the following commutative diagramm:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{M}^2 + I)/\mathcal{M}^2 & \longrightarrow & \mathcal{M}/\mathcal{M}^2 & \longrightarrow & \mathcal{M}/(\mathcal{M}^2 + I) & \longrightarrow & 0 \\
 & & \Big\downarrow a & & \Big\downarrow T(\widehat{f}) & & \Big\downarrow T(f) & & \\
0 & \longrightarrow & (\mathcal{M}^2 + J)/\mathcal{M}^2 & \longrightarrow & \mathcal{M}/\mathcal{M}^2 & \longrightarrow & \mathcal{M}/(\mathcal{M}^2 + J) & \longrightarrow & 0
\end{array}$$

where $a : (\mathcal{M}^2 + I)/\mathcal{M}^2 \to (\mathcal{M}^2 + J)/\mathcal{M}^2$ is the linear morphism sending the basis $\overline{u}_1, \ldots, \overline{u}_r$ of $(\mathcal{M}^2 + I)/\mathcal{M}^2$ on the basis $\overline{v}_1, \ldots, \overline{v}_r$ of $(\mathcal{M}^2 + J)/\mathcal{M}^2$. Therefore, a is a linear isomorphism. Furthermore, $T(f)$ is also a linear isomorphism (since f is an isomorphism). By the five's lemma, we can conclude that $T(\widehat{f})$ is a linear automorphism which shows that $\widehat{f}$ is an automorphism. $\square$

We will now prove three theorems which are easily deduced from lemma 2.1. The first will imply theorem A:

Theorem 2.1. If $N \geq 2$, any finite union of fat points is strongly analytically mono-embeddable in $\mathbb{A}^N$.

Proof. If $P^{[1]}, \ldots, P^{[m]}$ (resp. $Q^{[1]}, \ldots, Q^{[m]}$) are closed fat points of $\mathbb{A}^N$ with distinct supports and if $g^{[k]} : P^{[k]} \to Q^{[k]}$ is an isomorphism (for $1 \leq k \leq m$), then, by lemma 2.1, $g^{[k]}$ is induced by an analytic automorphism $f^{[k]}$ of $\mathbb{A}^N$. If $n \geq 1$, by theorem C of [6], there exists a (tame) analytic automorphism f such that the n-jets of f and $f^{[k]}$ coincide at the support of the closed fat point $P^{[k]}$ (for $1 \leq k \leq m$). If n has been chosen big enough, it is clear that f will extend each $g^{[k]}$. $\square$

If $u : \mathbb{A}^N \to \mathbb{A}^N$ is an analytic endomorphism of $\mathbb{A}^N$, let $u^\# : \widetilde{R} \to \widetilde{R}$, $r \mapsto r \circ u$ be the algebra-morphism induced by u. Let I, J be ideals of $\widetilde{R}$ of finite codimension. The last theorem implies that:

• Any algebra isomorphism $\widetilde{R}/I \to \widetilde{R}/J$ is induced by some $u^\#$, where u is an analytic automorphism of $\mathbb{A}^N$;

• The algebras $\widetilde{R}/I$ and $\widetilde{R}/J$ are isomorphic if and only if $u^\#(I) = J$ for some analytic automorphism u of $\mathbb{A}^N$.

The next theorem gives a sufficient condition in order that the algebra S_n/I does not admit any non-trivial $\mathbb{Z}$-grading. We begin with the:

Lemma 2.2. A finite complex algebra admits a non-trivial $\mathbb{Z}$-grading if and only if its automorphism group contains the torus $\mathbb{C}^*$.

Proof. Let B be a finite complex algebra. Its automorphism group $Aut\,B$ being closed in the linear group $GL(B)$, it is naturally an algebraic group. If $B = \bigoplus_k B_k$, then for each $t \in \mathbb{C}^*$, the map $\varphi_t \ : \ B \to B, \ \sum_k b_k \mapsto \sum_k t^k b_k$ is an algebra automorphism. Furthermore, if the grading is non-trivial, the group-morphism $\mathbb{C}^* \to Aut\,B,\ t \mapsto \varphi_t$ is injective. Conversely, if we have an injective morphism $\mathbb{C}^* \to Aut\,B,\ t \mapsto \varphi_t$, then $B = \bigoplus_k B_k$, where $B_k := \{b \in B,\ \forall\, t \in \mathbb{C}^*,\ \varphi_t(b) = t^k b\}$. It is a non-trivial grading. $\square$

If I is an ideal of S_n, its stabilizer is $Stab(I) := \{f \in Aut(S_n) = J_n(\widetilde{A}),\ f(I) = I\}$.

Theorem 2.2. If I is an ideal of S_n such that $Stab(I) \leq \{f \in J_n(\widetilde{A}),\ J_1 f = \mathrm{id}\}$, then S_n/I does not admit any non-trivial $\mathbb{Z}$-grading.

Proof. By lemma 1.1, we have $Stab(I) \leq R_u\left(J_n(\widetilde{A})\right)$ and by lemma 2.1, the natural map $Stab(I) \to Aut(S_n/I)$ is onto. Since any quotient and subgroup of a unipotent group is unipotent, $Aut(S_n/I)$ is unipotent. This shows that $Aut(S_n/I)$ does not contain any torus $\mathbb{C}^*$, so that we conclude by lemma 2.2. $\square$

We end with a useful criterion to decide whether the fat point $Spec\ S_n/I$ is algebraically mono-embeddable in $\mathbb{A}^N$ or not.

Theorem 2.3. If I is an ideal of S_n, the fat point $Spec\ S_n/I$ is algebraically mono-embeddable in $\mathbb{A}^N$ if and only if $J_n(\widetilde{A}) = J_n(A)\, Stab(I)$.

Proof. $Spec\ S_n/I$ is algebraically mono-embeddable in $\mathbb{A}^N$ if and only if for any ideal J of S_n such that $S_n/I \simeq S_n/J$ there exists $f \in J_n(A)$ such that $f(I) = J$. By lemma 2.1, we know that $S_n/I \simeq S_n/J$ if and only if there exists $g \in Aut(S_n) = J_n(\widetilde{A})$ such that $g(I) = J$. Therefore $Spec\ S_n/I$ is algebraically mono-embeddable in $\mathbb{A}^N$ if and only if

$$\forall\ f \in J_n(\widetilde{A}),\ \exists\ g \in J_n(A),\ f(I) = g(I).$$

By considering the action of $J_n(\widetilde{A})$ on the set of ideals of S_n, this can also be written $J_n(\widetilde{A}).I = J_n(A).I$ and this is equivalent to our wanted statement. $\square$

Corollary 2.1. If I is an ideal of S_n such that $Stab(I) \leq \{f \in J_n(\widetilde{A}),\ J_2 f = \mathrm{id}\}$ where $n \geq 2$, then $Spec\, S_n/I$ is not mono-embeddable in $\mathbb{A}^N$.

Proof. If $Spec\, S_n/I$ was mono-embeddable, we should have $J_n(\widetilde{A}) = J_n(A).Stab(I)$ and at the level of 2-jets we should have $J_2(\widetilde{A}) = J_2(A)$ which is not true. $\square$

If $H, K \leq G$, then K is called a complement of H in G if $G = HK$ and $H \cap K = \{1\}$ (see for example [11]). This is equivalent to: $\forall\ g \in G,\ \exists\,!\ (h,k) \in H \times K,\ g = hk$.

Corollary 2.2. If I is an ideal of S_n such that $Stab(I)$ contains a complement of $J_n(A)$

in $J_n(\widetilde{A})$, then $Spec\ S_n/I$ is mono-embeddable in $\mathbb{A}^N$.

III. COMPLEMENTS OF $J_n(A)$ IN $J_n(\widetilde{A})$.

In this section, we describe some nice complements of $J_n(A)$ in $J_n(\widetilde{A})$. In [6], we have seen that:

• $E_n = E_n^0 \oplus E_n^1$, where $E_n^0 := \{f \in E_n,\ \nabla f = 0\}$, $E_n^1 := \{p\,\mathrm{id},\ p \in R_{n-1}\}$, $\nabla f = \sum_L \dfrac{\partial f_L}{\partial x_L}$.

• If $j = \mathrm{id} + k \in J_n(E)$, where $n \geq 2$ and $k \in E_n$, then:
$$j \in J_n(A) \iff \mathrm{Jac}\ j = 1 \text{ in } J_{n-1}(R) \iff \nabla k = 0 \iff k \in E_n^0.$$

Therefore, if we set $M := \{p \in \widehat{R},\ p(0) = 1\}$ and $H := \{p \text{ id},\ p \in M\} \leq \widehat{A}$, the following result seems natural:

Proposition 3.1. For $n \geq 1$, $J_n(H)$ is a complement of $J_n(A)$ in $J_n(\widetilde{A})$.

Proof. By induction on n. For $n = 1$, it is clear since $J_1(A) = J_1(\widetilde{A}) = GL$ and $J_1(H) = \{\mathrm{id}\}$. Let us now prove the result for $n \geq 2$ assuming that it is true for $n-1$.

• Let us prove that $J_n(A) \cap J_n(H) = \{\mathrm{id}\}$.
If $j \in J_n(A) \cap J_n(H)$, then $J_{n-1}(j) \in J_{n-1}(A) \cap J_{n-1}(H) = \{\mathrm{id}\}$ by induction. Therefore, $j = \mathrm{id} + k$ where $k \in E_n$. Since $j \in J_n(A)$, we have $k \in E_n^0$ and since $j \in J_n(H)$, we have $k \in E_n^1$, so that $k = 0$.

• Let us prove that $J_n(\widetilde{A}) = J_n(A)J_n(H)$.
If $f \in \widetilde{A}$, by induction there exist $g \in A$ and $h \in H$ such that $J_{n-1}\, f = J_{n-1}\, g \circ J_{n-1}\, h$. Therefore, $J_{n-1}(g^{-1} \circ f \circ h^{-1}) = \mathrm{id}$, so that $J_n(g^{-1} \circ f \circ h^{-1}) = \mathrm{id} + k$ where $k \in E_n$. If $k = k_0 + k_1$, where $k_0 \in E_n^0$ and $k_1 \in E_n^1$, then $j_0 := \mathrm{id} + k_0 \in J_n(A)$ and $j_1 := \mathrm{id} + k_1 \in J_n(H)$, so that $J_n\, f = (J_n\, g \circ j_0) \circ (j_1 \circ J_n\, h) \in J_n(A)J_n(H)$. $\square$

We will now prove a more general version with weights. If $d = (d_1, \ldots, d_N) \neq 0 \in \mathbb{N}^N$, let us set $|d| := d_1 + \ldots + d_N$ and $H_d := \{(p^{d_1}x_1, \ldots, p^{d_N}x_N), p \in M\} \leq \widehat{A}$.

Lemma 3.1. If $p \in M$ and $h := (p^{d_1}x_1, \ldots, p^{d_N}x_N) \in H_d$, then
$$\mathrm{Jac}\ h = p^{|d|-1}\left(p + \sum_{1 \leq L \leq N} d_L\, x_L \frac{\partial p}{\partial x_L}\right).$$

Proof. The Jacobian matrix of h is equal to
$$h' = \begin{bmatrix} p^{d_1} & & 0 \\ & \ddots & \\ 0 & & p^{d_N} \end{bmatrix} + \begin{bmatrix} d_1\ x_1\ p^{d_1-1} \\ \vdots \\ d_N x_N p^{d_N-1} \end{bmatrix} . \left[\ \frac{\partial p}{\partial x_1},\ \ \ldots\ ,\ \ \frac{\partial p}{\partial x_N}\ \right].$$

Factorizing p^{d_L-1} on the L-th row, we get Jac $h = p^{|d|-N} \det V$ where

$$V := p\, I_N + \begin{bmatrix} d_1\ x_1 \\ \vdots \\ d_N x_N \end{bmatrix} . \left[\ \frac{\partial p}{\partial x_1}, \quad \dots \ , \quad \frac{\partial p}{\partial x_N} \ \right].$$

But if we set $U := -\begin{bmatrix} d_1\ x_1 \\ \vdots \\ d_N x_N \end{bmatrix} . \left[\ \frac{\partial p}{\partial x_1}, \quad \dots \ , \quad \frac{\partial p}{\partial x_N} \ \right]$, then rk $U \ \leq\ 1$ and Tr $U \ = \ -\sum_L d_L x_L \dfrac{\partial p}{\partial x_L}$, so that the characteristic polynomial of U is equal to

$$\chi_U(t) = \det(t\, I_N - U) = t^{N-1} \left(t + \sum_L d_L x_L \frac{\partial p}{\partial x_L} \right).$$

Therefore $\det V = \chi_U(p) = p^{N-1} \left(p + \sum_L d_L x_L \dfrac{\partial p}{\partial x_L} \right)$ and the resut follows. $\square$

Lemma 3.2. The map $\alpha \ : \ M \to M,\ p \mapsto p^{|d|-1} \left(p + \sum_{1 \leq L \leq N} d_L\, x_L \dfrac{\partial p}{\partial x_L} \right)$ is bijective.

Proof. We have $\alpha \ = \ \gamma \circ \beta$, where $\beta \ : \ M \ \to \ M,\ p \ \mapsto \ p^{|d|}$ and $\gamma \ : \ M \ \to \ M$ $q \mapsto q + \sum_{1 \leq L \leq N} \dfrac{d_L}{|d|}\, x_L \dfrac{\partial q}{\partial x_L}$. Let us check that β and γ are bijective.

• It is well known that β is bijective. This comes out from the fact that the map $(\mathbb{C}, 1) \to (\mathbb{C}, 1),\ x \mapsto x^{|d|}$ is a local analytic diffeomorphism.

• Let us note that $\gamma = \Gamma_{||M}$, where $\Gamma \ : \ \widehat{R} \to \widehat{R},\ q \mapsto q + \sum_{1 \leq L \leq N} \dfrac{d_L}{|d|}\, x_L \dfrac{\partial q}{\partial x_L}$ is a linear endomorphism satisfying: $\quad q \in M \iff \Gamma(q) \in M$.

Therefore, it is enough to show that Γ is bijective. But, for any $r \in \mathbb{N}^N$, we have $\Gamma(x^r) = \lambda_r x^r$, where $\lambda_r := 1 + \dfrac{1}{|d|} < d, r >$ with $< d, r > := \sum_{1 \leq L \leq N} d_L r_L$.

Since $\lambda_r \geq 1$, it is now clear that Γ is one-to-one: If $q = \sum_r q_r x^r \in \mathbb{C}\{x_1, \dots, x_N\}$, then the only possible preimage of q by Γ is $p := \sum_r \dfrac{q_r}{\lambda_r} x^r \in \mathbb{C}[[x_1, \dots, x_N]]$. It remains to show that $p \in \mathbb{C}\{x_1, \dots, x_N\}$. But if W is an open neighbourhood of the origin in $\mathbb{A}^N$ (for the transcendental topology) on which the series $q = \sum_r q_r x^r$ is normally convergent, then p will still be normally convergent on W since $\left| \dfrac{q_r}{\lambda_r} x^r \right| \ \leq \ |q_r x^r|$. $\square$

The next result is a consequence of the last two lemmas.

Lemma 3.3. The map $H_d \to M$, $h \mapsto \text{Jac } h$ is bijective.

Lemma 3.4. H_d is a complement of $K := \{f \in \widehat{A},\ \text{Jac } f \in \mathbb{C}^*\}$ in $\widehat{A}$.

Proof. If $f \in \widehat{A}$ and $h \in H_d$, then $\exists\ g \in K,\ f = g \circ h \Longleftrightarrow \text{Jac } h = \dfrac{\text{Jac } f}{(\text{Jac } f)(0)}$.

Therefore, the result follows from lemma 3.3. $\square$

Since $J_n(K) = J_n(A)$ and $J_n(\widehat{A}) = J_n(\widetilde{A})$, we get:

Proposition 3.2. If $n \geq 1$, then $J_n(H_d)$ is a complement of $J_n(A)$ in $J_n(\widetilde{A})$.

IV. PROOF OF THEOREM B.

Theorem 4.1. If $d = (d_1, \ldots, d_N) \neq 0 \in \mathbb{N}^N$, let us grade the algebra $R = \mathbb{C}[x_1, \ldots, x_N]$ by assigning each x_L to be homogeneous of degree d_L. If I is a homogeneous ideal of R such that R/I is a finite local algebra, then the fat point $Spec\, R/I$ is algebraically mono-embeddable in $\mathbb{A}^N$.

Proof. Let n be such that $(x_1, \ldots, x_N)^{n+1} \subset I$. Since $(x_1, \ldots, x_N)^{n+1}$ is a homogeneous ideal of R, the algebra $S_n := R/(x_1, \ldots, x_N)^{n+1}$ inherits a grading such that the canonical surjection $\pi\ :\ R \to S_n$ is a graded morphism. If we set $\overline{I} = \pi(I)$, then $R/I \simeq S_n/\overline{I}$. But $J_n(H_d) \leq Stab(\overline{I})$ and $J_n(H_d)$ is a complement of $J_n(A)$ in $J_n(\widetilde{A})$ by proposition 3.2. The result follows from cor. 2.2. $\square$

Corollary 4.1. If X is a fat point such that $ed(X) < N - 1$, then X is algebraically mono-embeddable in $\mathbb{A}^N$.

Proof. There exists an ideal J of $\mathbb{C}[x_1, \ldots, x_{N-1}]$ such that if $I := J.R + x_N.R$, then the algebra of functions of X is isomorphic to R/I. If we endow R with the grading where $x_1, \ldots, x_{N-1}$ are homogeneous of degree 0 and x_N is homogeneous of degree 1, it is enough to note that I is homogeneous. $\square$

Proof of theorem B. Let $B = \bigoplus_{k \geq 0} B_k$ be a finite local complex algebra endowed with a non-trivial $\mathbb{Z}_+$-grading. If $\mathcal{N}$ is the maximal ideal of B, let us begin to show that $\mathcal{N}$ is homogeneous. If $b = \sum_{k \geq 0} b_k \in \mathcal{N}$, $b_k \in B_k$, we want to show that $b_k \in \mathcal{N}$. If $k \geq 1$, it is clear, since b_k is nilpotent. Therefore $b_0 = b - \sum_{k \geq 1} b_k \in \mathcal{N}$ also. Let $h_1, \ldots, h_m$ be a homogeneous basis of the vector space $\mathcal{N}$. The family $\overline{h}_1, \ldots, \overline{h}_m$ generates the vector space $\mathcal{N}/\mathcal{N}^2$ so that we can extract from it a basis of $\mathcal{N}/\mathcal{N}^2$. We may assume that $ed(B) = N$. Indeed, if $ed(B) < N$, we have already seen that *Spec* B is mono-embeddable in $\mathbb{A}^N$ and if $ed(B) > N$, then *Spec* B is clearly mono-embeddable in $\mathbb{A}^N$ since it cannot be embedded in $\mathbb{A}^N$, which shows that all its embeddings are equivalent ! Since we have found homogeneous elements $u_1, \ldots, u_N$ of B which generate the algebra B, this shows that B can be written as in theorem 4.1. $\square$

V. INTERPOLATION AND RIGIDITY LEMMAS.

We set $N = 2$, so that $R = \mathbb{C}[x, y]$. We recall that R_k denotes the set of k-homogeneous polynomials of R. We will denote by $\mathbb{C}_k$ the space of complex polynomials in the indeterminate x whose degree is less than or equal to k. The three following rigidity lemmas will be used in the next section.

Lemma 5.1 (first rigidity lemma). If $l_1, \ldots, l_m \in R_1$ are linearly independent, then for any integers $k \geq 0$, $d \geq (m-1)(k+1)$, the following map is injective

$$\begin{array}{rlcl} \varphi : & (R_k)^m & \to & R_{k+d} \\ & (r_i)_{1 \leq i \leq m} & \mapsto & \displaystyle\sum_i r_i \, (l_i)^d. \end{array}$$

Proof. We may assume that $l_i = x + \lambda_i y$, where the λ_i are distinct complex numbers.

Setting $y = 1$, it is enough to show that the following map is injective

$$\begin{array}{rlcl} \Phi : & (\,\mathbb{C}_k\,)^m & \to & \mathbb{C}_{k+d} \\ & (r_i)_{1 \leq i \leq m} & \mapsto & \displaystyle\sum_i r_i \, (x + \lambda_i)^d. \end{array}$$

Since for each i, the family $(x + \lambda_i)^j$, $0 \leq j \leq k$ is a basis of $\mathbb{C}_k$, this amounts to show that the family $(x + \lambda_i)^{d+j}$, $1 \leq i \leq m$, $0 \leq j \leq k$ is linearly independent.

By derivating with respect to x, it is enough to show the same result where d is replaced by $d - 1$. Therefore, we may assume that $d = (m-1)(k+1)$.

Setting $n := k + 1$, we want to show that the family $(x + \lambda_i)^{mn-j}$, $1 \leq i \leq m$, $1 \leq j \leq n$ is linearly independent. But since $\dim \mathbb{C}_{mn-1} = mn$, we will in fact show that it is a basis of $\mathbb{C}_{mn-1}$, which comes from the next more general result. $\square$

Proposition 5.1. Let $\lambda_1, \ldots, \lambda_m$ be distinct complex numbers and let $w_1, \ldots, w_m$ be nonnegative integers. If we set $w := \displaystyle\sum_i w_i$, then the family $(x + \lambda_i)^{w-j}$, $1 \leq i \leq m$, $1 \leq j \leq w_i$ is a basis of $\mathbb{C}_{w-1}$.

Proof. In fact, this result is a consequence of the Hermite's interpolation theorem asserting that given any complex numbers $\beta_{i,j}$ there exists a unique polynomial $p \in \mathbb{C}_{w-1}$ satisfying $p^{(j-1)}(\lambda_i) = \beta_{i,j}$ for $1 \leq i \leq m$, $1 \leq j \leq w_i$.

Writting $p = \sum_{1\le a\le w} p_a x^{a-1}$, this is equivalent to $\sum_{1\le a\le w} p_a \binom{a-1}{j-1}\lambda_i^{a-j} = \frac{1}{(j-1)!}\beta_{i,j}$, where we agree that $\binom{a}{b} = 0$ if we do not have $0 \le b \le a$.

If $1 \le i \le m$, let $P_i \in M_{w,w_i}(\mathbb{C})$ be the matrix defined by its general term

$$(P_i)_{a,b} = \binom{a-1}{b-1}\lambda_i^{a-b},\ 1 \le a \le w,\ 1 \le b \le w_i.$$

If $P := [P_1, P_2, \ldots, P_m] \in M_w(\mathbb{C})$, then P is invertible by Hermite's theorem.

However, by multiplying the a-th row of P_i by $\binom{w-1}{a-1}$ and by dividing the b-th column by $\binom{w-1}{b-1}$, we obtain the matrix $Q_i \in M_{w,w_i}(\mathbb{C})$ with general term

$$(Q_i)_{a,b} = \binom{a-1}{b-1}\lambda_i^{a-b} \times \frac{\binom{w-1}{a-1}}{\binom{w-1}{b-1}} = \binom{w-b}{w-a}\lambda_i^{a-b},\ 1 \le a \le w,\ 1 \le b \le w_i.$$

Therefore, the matrix $Q = [Q_1, \ldots, Q_m] \in M_w(\mathbb{C})$ is invertible.

But each Q_i is the matrix of the family $(x+\lambda_i)^{w-1}, \ldots, (x+\lambda_i)^{w-w_i}$ expressed in the canonical basis $(x^{w-1}, x^{w-2}, \ldots, x, 1)$ of $\mathbb{C}_{w-1}$. The invertibility of Q exactly means that the family $(x+\lambda_i)^{w-j}$, $1 \le i \le m$, $1 \le j \le w_i$ is a basis of $\mathbb{C}_{w-1}$. $\square$

Remark. The above matrix P associated with the Hermite's interpolation problem has been very often introduced in the literature (see for example [8], [15]) and is a generalization of the Vandermonde matrix. It is for example shown in [13] that

$$\det P = \prod_{1\le i<j\le m} (\lambda_j - \lambda_i)^{w_i w_j}.$$

The proof is by reverse induction on the number m of blocks (with w fixed) beginning with the usual Vandermonde matrix for $m = w$. Using this result, we obtain at once

$$\det Q = \frac{\prod_{1\le a\le w}\binom{w-1}{a-1}}{\prod_{1\le i\le m}\prod_{1\le b\le w_i}\binom{w-1}{b-1}} \prod_{1\le i<j\le m}(\lambda_j - \lambda_i)^{w_i w_j} \neq 0.$$

Lemma 5.2 (second rigidity lemma). If $p, q \in R_n$, the following assertions are equivalent:

(i) there exists $h \in R_{n-1}$ such that $p = xh$, $q = yh$;
(ii) for any $\lambda \in \mathbb{P}^1$, $x + \lambda y$ divides $p + \lambda q$;
(iii) for at least $n+2$ values of $\lambda \in \mathbb{P}^1$, $x + \lambda y$ divides $p + \lambda q$.

Remark. For $\lambda = \infty$, the relation $x + \lambda y$ divides $p + \lambda q$ means that y divides q.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i). We may assume that $(p,q) \neq (0,0)$. Then (p,q) induces a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\deg f \leq n$. However, (iii) means that f admits at least $n+2$ fixed points, which implies $f = \mathrm{id}_{\mathbb{P}^1}$. $\square$

If $a_1, \ldots, a_4$ are 4 distinct points of $\mathbb{P}^1$, let us recall that their cross-ratio is defined by $[a_1, a_2, a_3, a_4] = \frac{a_3-a_1}{a_3-a_2} / \frac{a_4-a_1}{a_4-a_2} \in \mathbb{P}^1$. If $b_1, \ldots, b_4$ are 4 distinct points of $\mathbb{P}^1$, it is well known that there exists a homography of $\mathbb{P}^1$ sending a_k on b_k for $1 \leq k \leq 4$ if and only if $[a_1, a_2, a_3, a_4] = [b_1, b_2, b_3, b_4]$. If we permute the a_k, the cross-ratio $\lambda = [a_1, a_2, a_3, a_4]$ may change, but not the expression $\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}$. Therefore, one usually defines the j-invariant of $\{a_1, \ldots, a_4\}$ by this formula. Furthermore, there exists a homography of $\mathbb{P}^1$ sending $\{a_1, \ldots, a_4\}$ on $\{b_1, \ldots, b_4\}$ if and only if $j(\{a_1, \ldots, a_4\}) = j(\{b_1, \ldots, b_4\})$ (see the definition of the j-invariant of an elliptic curve in § IV.4 of [7] or § 6.3.3 of [4]). If X is any set, let us denote by $\mathcal{P}_4(X)$ the set of all subsets of X with exactly 4 elements.

Definition 5.1. We will say that $X \subset \mathbb{P}^1$ is j-separated if $j : \mathcal{P}_4(X) \to \mathbb{P}^1$ is injective.

The next result is almost obvious.

Lemma 5.3 (third rigidity lemma). If $\mathcal{A}$ is a j-separated subset of $\mathbb{P}^1$ with at least 5 elements and if h is a homography of $\mathbb{P}^1$ satisfying $h(\mathcal{A}) = \mathcal{A}$, then $h = \mathrm{id}_{\mathbb{P}^1}$.

Proof. Let $a_1, \ldots, a_5$ be five distinct points of $\mathcal{A}$. If we set $\mathcal{A}_i := \{a_1, \ldots, a_5\} \setminus \{a_i\}$, we must have $h(\mathcal{A}_i) = \mathcal{A}_i$ thanks to the j-separatedness. This clearly implies $h(a_i) = a_i$ and since h fixes at least 3 points $h = \mathrm{id}_{\mathbb{P}^1}$. $\square$

Lemma 5.4 (adjunction lemma). If $\mathcal{A}$ is a finite j-separated subset of $\mathbb{P}^1$ and if $\mathcal{B}$ is an infinite subset of $\mathbb{P}^1$, then there exists $b \in \mathcal{B}$ such that $\mathcal{A}' := \mathcal{A} \cup \{b\}$ is j-separated.

Proof. If $f \in \mathbb{C}(x)$ is a non constant rational function and if $c \in \mathbb{P}^1$, then there exist only finitely many $b \in \mathbb{P}^1$ such that $f(b) = c$. $\square$

Using this adjunction lemma, we can make the following

Definition 5.2. We define the sequence $(\alpha_n)_{n \in \mathbb{N}}$ inductively by $\alpha_0 = 0$ and for each $n \geq 1$, α_n is the least integer such that

(i) $\alpha_n > \alpha_{n-1}$ and (ii) $\{\alpha_0, \ldots, \alpha_n\}$ is j-separated.

With the help of a computer, one finds easily $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 5$, $\alpha_5 = 12$, $\alpha_6 = 15$, $\alpha_7 = 32$, $\alpha_8 = 38$, $\alpha_9 = 43$, $\alpha_{10} = 58$.

VI. PROOF OF THEOREM C.

In this section, we still set $N = 2$ so that $S_n = \mathbb{C}[x,y]/(x,y)^{n+1}$. We will give examples of fat points with embedding dimensions 2, without any non-trivial $\mathbb{Z}_+$-grading, which are (resp. which are not) algebraically mono-embeddable in $\mathbb{A}^2$. The α_i used in the next result are given in definition 5.2 above.

Theorem 6.1. If $k \geq 1$, let m, d, n be such that $m \geq \max(k+1, 4)$, $d \geq (m+1)(k+2)+1$, $n \geq d+k$. If I is the ideal of S_n generated by $x^d - y^{d+1}$ and $(x-\alpha_i y)^d$, $1 \leq i \leq m$, then $Stab(I) \leq \{f \in J_n(\widetilde{A}),\ J_k f = \mathrm{id}\}$.

Remark. Roughly speaking

- the homogeneous elements $(x-\alpha_i y)^d$, $1 \leq i \leq m$, imply that $J_k\, f$ is a generalized dilatation, i.e. $J_k\, f = \lambda\, \mathrm{id}$ where $\lambda \in R$ satisfies $\deg \lambda \leq k-1$;
- the non homogeneous element $x^d - y^{d+1}$ implies $\lambda = 1$.

Proof. If f fixes I, it also fixes $I + (x,y)^{d+k+1}$. Therefore, we may assume that $n = d+k$.

If $p \in S_n$, let us define its initial term, denoted $in(p)$, as its homogeneous term of smallest degree. We define $in(I)$ as the (homogeneous) ideal of S_n generated by the $in(p)$, $p \in I$. For $0 \leq l \leq k$, we have $d \geq m(k+1) \geq m(l+1)$. Using lemma 5.1 with the $m+1$ linear forms $x - \alpha_i y$, $0 \leq i \leq m$, the map

$$\begin{array}{cccl} \varphi : & (R_l)^{m+1} & \to & R_{l+d} \\ & (r_i)_{0 \leq i \leq m} & \mapsto & \displaystyle\sum_{0 \leq i \leq m} r_i\, (x-\alpha_i y)^d \end{array}$$

is injective. This shows that the ideal $in(I)$ is generated by the $(x-\alpha_i y)^d$, $0 \leq i \leq m$.

First step. Let us show that the linear part of f is a dilatation, i.e. $\mathcal{L}(f) = \lambda.\mathrm{id}$, where $\lambda \in \mathbb{C}^*$.

Since $f(x^d - y^{d+1}) \in I$ and since $in\big(f(x^d - y^{d+1})\big) = (\mathcal{L}(f)(x))^d$, we get $(\mathcal{L}(f)(x))^d \in in(I)$. By the same way, we could show that $(\mathcal{L}(f)(x-\alpha_i y))^d \in in(I)$, for $0 \leq i \leq m$. Let h be the homography of $\mathbb{P}^1$ induced by the invertible linear automorphism $\mathcal{L}(f)$. Since h fixes $\{\alpha_0, \ldots, \alpha_m\}$ and since $m \geq 4$, using lemma 5.3, we get $h = \mathrm{id}_{\mathbb{P}^1}$.

Second step. Let us show that $J_1(f) = \mathrm{id}$.

By the first step, there exist $\lambda \in \mathbb{C}^*$ and $p \in R_2$ such that $f_1 \equiv \lambda x + p \mod \mathcal{M}^3$ and $f_2 \equiv \lambda y \mod \mathcal{M}^2$. This implies that
$\frac{1}{\lambda^d} f(x^d - y^{d+1}) - (x^d - y^{d+1}) \equiv \frac{d}{\lambda} p x^{d-1} + (1-\lambda) y^{d+1} \mod \mathcal{M}^{d+2}$.

Since $\frac{1}{\lambda^d} f(x^d - y^{d+1}) - (x^d - y^{d+1}) \in I$ and since $\frac{d}{\lambda} p x^{d-1} + (1-\lambda) y^{d+1} \in R_{d+1}$, we get $\frac{d}{\lambda} p x^{d-1} + (1-\lambda) y^{d+1} \in in(I)$, so that there exist $a_0, \ldots, a_m \in R_2$ such that $(1-\lambda) y^2 y^{d-1} + \displaystyle\sum_{0 \leq i \leq m} a_i\, (x-\alpha_i y)^{d-1} = 0$.

Since $d-1 \geq 3(m+1)$, using lemma 5.1 with the $m+2$ linear forms $(x-\alpha_i y)_{0 \leq i \leq m}$ and y, we get $(1-\lambda) y^2 = a_0 = \ldots = a_m = 0$, whence $\lambda = 1$.

If $k = 1$, the theorem is proven. If $k \geq 2$, let l be an integer such that $2 \leq l \leq k$ and let us assume that $f_1 \equiv x + p \mod \mathcal{M}^{l+1}$, $f_2 \equiv y + q \mod \mathcal{M}^{l+1}$ where $p, q \in R_l$. It is enough to prove that $p = q = 0$.

Third step. Let us show that there exists $h \in R_{l-1}$ such that $p = xh$, $q = xh$.

If $1 \leq i \leq m$, we have $f\left((x-\alpha_i y)^d\right) - (x-\alpha_i y)^d \equiv [(x-\alpha_i y)+(p-\alpha_i q)]^d - (x-\alpha_i y)^d \equiv d(p-\alpha_i q)(x-\alpha_i y)^{d-1} \mod \mathcal{M}^{d+l}$, so that $(p-\alpha_i q)(x-\alpha_i y)^{d-1} \in in(I)$.

Therefore, there exist $a_0, \ldots, a_m \in R_{l-1}$ such that

$$(p-\alpha_i q)(x-\alpha_i y)^{d-1} = \sum_{0 \leq j \leq m} a_j (x-\alpha_j y)^d.$$

Since $d - 1 \geq m(k+1) \geq m(l+1)$, using lemma 5.1 with the $m+1$ linear forms $(x-\alpha_j y)_{0 \leq j \leq m}$, we get $p - \alpha_i q = a_i(x-\alpha_i y)$, so that $x - \alpha_i y$ divides $p - \alpha_i q$.

If $i = 0$, we have $f(x^d - y^{d+1}) - (x^d - y^{d+1}) \equiv dpx^{d-1} \mod \mathcal{M}^{d+l}$ so that x divides p by the same way.

Therefore, $x - \alpha_i y$ divides $p - \alpha_i q$ for $0 \leq i \leq m$ and since $m + 1 \geq k + 2 \geq l + 2$, we are done by lemma 5.2.

Fourth step. Let us show that $h = 0$.

We have $f_1 \equiv x + xh \mod \mathcal{M}^{l+1}$, $f_2 \equiv y + yh \mod \mathcal{M}^{l+1}$, where $h \in R_{l-1}$, $2 \leq l \leq k$.

Let us note that $1 + h$ is invertible in S_n and that $\frac{f_1}{1+h} \equiv x \mod \mathcal{M}^{l+1}$. Let $r \in R_{l+1}$ be such that $\frac{f_1}{1+h} \equiv x + r \mod \mathcal{M}^{l+2}$.

We have $\frac{1}{(1+h)^d} f(x^d - y^{d+1}) - (x^d - y^{d+1}) \equiv (x+r)^d - (1+h)y^{d+1} - (x^d - y^{d+1}) \equiv drx^{d-1} - hy^{d+1} \mod \mathcal{M}^{d+l+1}$.

Since $\frac{1}{(1+h)^d} f(x^d - y^{d+1}) - (x^d - y^{d+1}) \in I$ and $drx^{d-1} - hy^{d+1} \in R_{d+l}$, we get $drx^{d-1} - hy^{d+1} \in in(I)$.

Therefore, there exist $a_0, \ldots, a_m \in R_{l+1}$ such that $hy^2 y^{d-1} + \sum_{0 \leq i \leq m} a_i (x-\alpha_i y)^{d-1} = 0$.

Since $d - 1 \geq (m+1)(k+2) \geq (m+1)(l+2)$, by lemma 5.1 applied with the $m+2$ linear forms $(x-\alpha_i y)_{0 \leq i \leq m}$ and y, we get $hy^2 = a_0 = \ldots = a_m = 0$, so that $h = 0$. $\square$

Corollary 6.1. Let I be the ideal of S_{17} generated by $x^{16} - y^{17}$ and $(x - \alpha y)^{16}$, $\alpha \in \{1, 2, 3, 5\}$, then S_{17}/I does not admit any non trivial $\mathbb{Z}$-grading, but $Spec\, S_{17}/I$ is algebraically mono-embeddable in $\mathbb{A}^2$.

Proof. We have $J_{17}(H) \leq Stab(I) \leq \{f \in J_{17}(\widetilde{A}),\, J_1(f) = \mathrm{id}\}$. Since $J_{17}(H)$ is a complement of $J_{17}(A)$ in $J_{17}(\widetilde{A})$ by proposition 3.1, the first inclusion shows us that $Spec\, S_{17}/I$ is mono-embeddable in $\mathbb{A}^2$ (see cor. 2.2). The second inclusion shows us that S_{17}/I does not admit any non-trivial $\mathbb{Z}$-grading (see theorem 2.2). $\square$

Remark. We may of course find some "smaller" examples. We leave as an exercise to the reader the fact that if I is the ideal of S_8 generated by $x^7 - y^8, x^3y^4, (x+y)^7$, then S_8/I does not admit any non-trivial $\mathbb{Z}$-grading, but $Spec\, S_8/I$ is algebraically mono-embeddable in $\mathbb{A}^2$.

Corollary 6.2. Let I be the ideal of S_{23} generated by $x^{21} - y^{22}$ and $(x - \alpha y)^{21}$, $\alpha \in \{1, 2, 3, 5\}$, then $Spec\, S_{23}/I$ is not algebraically mono-embeddable in $\mathbb{A}^2$.

Proof. We have $Stab(I) \leq \{f \in J_{23}(\widetilde{A}),\ J_2(f) = \mathrm{id}\}$ by theorem 6.1. The result follows from cor. 2.1. $\square$

VII. APPENDIX.

If X is a fat point such that $ed(X) < N$, it is quite easy to show that X is strongly algebraically mono-embeddable in $\mathbb{A}^N$. Indeed, by cor. 4.1, X is algebraically mono-embeddable in $\mathbb{A}^N$. Therefore, we may assume that X is embedded in $\mathbb{A}^{N-1} \subset \mathbb{A}^N$, where $\mathbb{A}^{N-1} = \{(x_1, \ldots, x_N) \in \mathbb{A}^n,\ x_N = 0\}$, and it is sufficient to show that any automorphism $f\ :\ X \to X$ can be extended in an algebraic automorphism $\beta\ :\ \mathbb{A}^N \to \mathbb{A}^N$. We may also assume that X is embedded in $Y = Spec\ \mathbb{C}[x_1, \ldots, x_{N-1}]/(x_1, \ldots, x_{N-1})^{n+1} \subset \mathbb{A}^{N-1}$ for some n. By lemma 2.1, f is induced by some automorphism $\widehat{f}$ of Y. By theorem 6.1 of [6], $\widehat{f}$ is induced by some algebraic automorphism β of $\mathbb{A}^N$. By the same way, we could even show that if X is any finite union of fat points such that $ed(X) < N$, then X is strongly algebraically mono-embeddable in $\mathbb{A}^N$.

However, it is easy to find a fat point which is not strongly algebraically mono-embeddable in $\mathbb{A}^N$. If we set $X := Spec\ \mathbb{C}[x_1, \ldots, x_N]/(x_1, \ldots, x_N)^3$ and if we consider $f := (x_1 + x_1^2, x_2, \ldots, x_N) \in Aut(X) \simeq J_2(\widetilde{A})$, then f cannot be extended in a polynomial automorphism of $\mathbb{A}^N$, since Jac $f = 1 + 2x_1 \in J_1(R)$ is not a constant.

We end with a few comments about the algebraic mono-embeddability of finite unions of fat points. Let $X = \underset{1 \leq k \leq m}{\cup} X^{[k]}$, where the $X^{[k]}$s are distinct fat points. We leave as an easy exercise for the reader, the fact that if X is algebraically mono-embeddable in $\mathbb{A}^N$, then the $X^{[k]}$s also. Unfortunately, the converse is not true. Indeed, if I is the ideal of S_{17} given in cor. 6.1, we have seen that any $f \in Stab(I)$ satisfies $J_1 f = \mathrm{id}$ and that the fat point $F := Spec\, S_{17}/I$ is algebraically mono-embeddable in $\mathbb{A}^2$. However, the disjoint union of two copies of F is not algebraically mono-embeddable in $\mathbb{A}^2$. Indeed, let P be a closed fat point of $\mathbb{A}^2$ isomorphic to F and whose support is at the origin of $\mathbb{A}^2$. Let $Aut(\mathbb{A}^2)$ denote the group of algebraic automorphisms of $\mathbb{A}^2$. Let $h := (2x, 2y) \in Aut(\mathbb{A}^2)$ be the dilatation of ratio 2 and let $\tau := (x+1, y) \in Aut(\mathbb{A}^2)$ be the translation of vector $(1, 0)$. Then, the closed subschemes $X_1 := F \cup \tau(F)$ and $X_2 := h(F) \cup \tau(F)$ of $\mathbb{A}^2$ are both isomorphic to X, but there does not exist any $f \in Aut(\mathbb{A}^2)$ such that $f(X_1) = X_2$: such an f should both satisfy $\mathrm{Jac}\, f = 1$ and $\mathrm{Jac}\, f = 4$.

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