Putting Non Convex Interval Mutual Relation Models into Practice

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Abstract

J.F. Allen's work provides an essential theoretical framework for convex intervals calculus. G. Ligozat extended this fundamental work to the case of non convex intervals. Most of the algebraic properties are preserved. However, the lattice of the relation between non convex intervals is too important for being mastered and used as a whole in practical industrial applications. Within this respect, we select a set of relevant extended relation types(namely ALLEN) which can apply to non convex intervals while bearing a semantics close to the genuine Allen's one. We provide a formal specification of these relation types and develop a composition calculus that results in transitivity tables. The paper places our work within Ligozat's theory and special properties are enounced about the set ALLEN*.*
As a conclusion, we evoke practical use cases for our proposal which has been experimented on the occasion of an ANR project.

1 Introduction

The seminal Allen's framework deals with relations between convex intervals [Allen81], [Allen83] and proves to be most efficient within the field of temporal data processing. This original proposal has led to various important developments, namely extending the scope to point and non-convex interval patterns [Vila82] [Ligo91].
Within this theoretical framework, it is therefore possible to address the rigorous specification and processing of temporal information including repeated events considered either with or without duration. With respect to these goals, P. Terenziani [Tere02] presents his own proposal, backed up with a comprehensive review of constructive formal languages dedicated to expressing both user-defined periodicity and temporal constraints about recurrent events.
Furthermore, B. Leban and al. [Leba86] such as M. Niezette and al. [Niez92], provide grammars and operators permitting – among other things - to express the semantics of calendars. Consequently, the calendar semantics can be turned to good account by bridging qualitative reasoning about relations between points or intervals, with geometric and topologic knowledge about date-time stamped events.

This paper is motivated by the need of specifying a body of concepts and rules that could aid end-users interacting with event databases management systems. We have in mind such kinds of applications which involve temporal data in Information Systems (e.g.: recording, verifying, querying, transforming…), for instance when managing access periods to social events (culture, tourism, accommodation…), availability of various resources (servers, devices, after-sales service …), concurrent iterated tasks scheduling, etc.

Despite the clear and complete algebraic, combinatorial and logical structures of the set of qualitative relation between non-convex intervals evoked above, there are very few studies coping with the "putting into practice" side of the matter. Therefore, in contrast with the thirteen Allen binary relations between convex intervals which bear relevant semantics, there is an actual lack of abstraction when dealing with non convex intervals. In that case, the set of binary relations presents with only few elements of practical interest, i.e.: relations which can make sense in everyday life. P. Ladkin [Ladk87] has listed relations of specific relevance for specifying some aspects of concurrent process behaviour, but – as far as we know – aside from domain specific approaches, no general purpose study has been performed yet.

Here, we intend to specify a kernel of relations types between non-convex intervals whose element are sound in terms of common sense knowledge, i.e. for instance, temporal event occurrence management whatever the application domain. Of course, our proposal is inspired by Allen's genuine relations which constitute a firm foundation; and we then use Ligozat's extended relation theoretical background to investigate the basic properties of the kernel we specify.

The paper first recalls Allen's and Ligozat's formalisms in Section 2 and Section 3 respectively. The selected meaningful relation types for non-convex intervals constituting the kernel are listed in the first part of Section 4 which is twofold. Its second sub-section is devoted to presenting the main properties of the kernel. We sketch illustrative examples of industrial applications of our proposal in Section 5, before concluding with Section 6 and finally providing references in Section 7.

2 Allen's Interval Algebra

This section briefly recalls the main features – definitions and properties - of Allen's Interval Algebra. Here, we set the shortcut notation we shall use all along this paper for naming Allen's relations. At the end of the section, we introduce a coding rule which is the basis of Ligozat's extension of Allen's proposal to non-convex intervals.

2.1 Definitions

In the sense of Allen a convex interval is uniquely defined by an ordered pair of distinct elements in the domain ($\mathcal{D}$) of a linear order ($\leq$); hence a possible straightforward implementation within the field of temporal intervals.

More precisely, a given interval I is equivalent to a pair (a, b) such that:

$(a, b \in \mathcal{D} \ \wedge \ a < b)$ *in all the following,* $(a<b)$ *is a surrogate for* $(a \leq b \in a \neq b)$

Then: $I = \{ x \in \mathcal{D} \mid a \leq x \ \wedge \ x \leq b \}$

a and b respectively are the lower and upper bounds of the interval I = (a, b)

Let $I\,[\mathcal{D}, \leq]$ be the set of such intervals.

Combining all possible relative positions of their bounds (with respect to the order $\leq$, i.e.: $<, =, >$) Allens specifies the 13 well known kinds of elementary mutual relations between two convex intervals.

2.2 Terminology and Notation

Many abbreviation systems may be found in the literature to denote the 13 relations. We adopt the following:

precedes	(p)	(P)	Preceded by
meets	(m)	(M)	Met by
overlaps	(o)	(O)	Overlapped by
finishes	(f)	(F)	Finished by
during	(d)	(D)	Contains
begins	(b)	(B)	Begun_by

equals (e)

Each relation is defined by constraints (involving the order <) upon the bounds of both intervals in hand.
Then, let ALLEN = { p, P, m, M, o, O, f, F, d, D, b, B, e} be the set of all possible relations.
Except for relation "equals" which is reflexive, the low case and upper case respectively indicate one relation and its inverse.

2.3 Canonical representation

The mutual position of two intervals can be fairly and concisely represented through mapping the set of interval bounds to the set of natural numbers as presented below.
Let X = (x1, x2) and Y = (y1, y2), be two intervals verifying X <u>R</u> Y where R $\in$ ALLEN.
Let us assume that the bounds of Y are mapped to the odd numbers 1 and 3. This results in defining five adjacent areas including singletons {T0 ,… , T4}, namely:

T0 $=\{t \in \mathcal{D} \mid t < y1\}$ T1 = {y1} T2 $= \{t \in \mathcal{D} \mid y1 < t \wedge t < y2\}$,
T3 = {y2} T4 $= \{t \in \mathcal{D} \mid y2 < t\}$

Any relational pattern between X and Y can be represented by the sequence of the two ranks of the areas where x1 and x2 respectively lay.

For instance, *precedes* is coded by (0,0), *Preceded_by* corresponds to (4,4), (1,2) is for *begins* and (2,3) means *Begun_by* …Let us notice that the equality of the two elements in the sequence can only occur for even numbers and in no way for odd ones, since empty intervals are not admitted.

2.4 Allen's Algebra main properties

In this section, we shall distinguish between set theoretics, combinatorial and relational aspects of Allen's Algebra properties.

2.4.1 Set theoretics

Considering any relation in ALLEN as an element in POW ($I[\mathcal{D}, \leq]$ x $I[\mathcal{D}, \leq]$), The 13 Allen's basic relations define a partition; i.e.: any ordered pair of intervals taken from $I[\mathcal{D}, \leq]$ satisfies one and only one basic relation in ALLEN.

Besides, ALLEN generates an algebra, namely POW(ALLEN), and various predicates can be defined as disjunctions of ALLEN's atomic elements, and account for non elementary qualitative meaningful relationships such as "intercept", "adjacent", etc.

2.4.2 Combinatorial properties

A precedence relation can be defined upon ALLEN, hence providing it with a distributive lattice structure. The canonical representation given in 2.3 is much appropriate for building a definition of the lattice, on which further definitions can be backed up.

Starting from a given configuration between two intervals X and Y, it clearly appears that – provided it is permitted - moving one bound of X leftward or rightward (i.e.: from the original area where it lays to the previous or next one), necessarily modifies the mutual relation between X and Y. On this basis, it is easy to observe that a topology can be defined on the various configurations of two intervals, say: X=(x1, x2) and Y=(y1,y2). Here, the "one step" bound movement corresponds to a kind of continuity.

In fact, let R belong to ALLEN. R has a canonical representation (i,j), with:

$$(i, j \in 0..4 \;\wedge\; i \leq j \;\wedge\; (i = j \Rightarrow \text{even}(i))).$$

When ($i<3 \wedge j>1$), R has two direct neighbours, namely (i, j+1) and (i+1, j). It has only one otherwise (see Fig. 1).

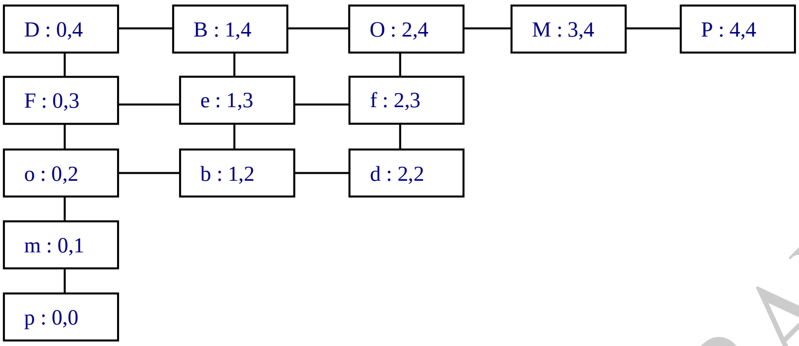


Fig.1 : Allen's Interval Mutual Relation Lattice Structure (Hasse Diagram)

Obviously, the order ($\prec$) underlying the distributive lattice structure is the product order on each coordinate of the canonical representation:

Le's assume X = (ix, jx) $\wedge$ Y = (iy, jy) then: (X $\prec$ Y $\Leftrightarrow$ ix $\leq$ iy $\wedge$ jx $\leq$ jy)

2.4.3 Relational issues

If considered as a set of binary relations, ALLEN is stable for relational inversion and composition.

Inversion is straightforward and is taken into account *via* low and upper cases.

Studying the composition (denoted by ⨾) of two relations leads to building transitivity tables which are most useful in reasoning.

Composition is defined as follows:

$$\forall\ R1, R2, R3 \in ALLEN \mid R3 = R1 \fatsemi R2 \;\bullet$$

$$R3 = \{\ X,Y,Z : I\,[\mathcal{D}, \leq\,] \mid X\ \underline{R1}\,Y \wedge Y\ \underline{R2}\ Z \;\bullet\; (X, Z)\ \}$$

One important property of the composition is its indeterminacy, since R ⨾ S may depend on the special geometric configuration (non qualitative) of the two intervals in hand.

For instance, from (X <u>B</u> Y ∧ Y <u>m</u> Z), one can deduce but (X <u>D</u> Z or X <u>F</u> Z or X <u>o</u> Z).

Conversely, (X <u>s</u> Y ∧ Y <u>m</u> Z) implies X <u>p</u> Z.

Nevertheless, the set of all possible results for one given composition is strictly constrained and necessarily constitute an interval[1] in the lattice.

The transitivity properties are the basis of computation about satisfiability of a set of constraints upon Allen's intervals configuration. This topic is addressed by many studies within the domain of SAT problems, but actually is out of the scope of this paper.

Table 2. expresses the transitivity of Allen's relations. For sake of conciseness, only the boundaries of the lattice intervals have been mentioned. The full interval can be read from Fig. 1.

Cells with a dark background are singletons.

[1] For any two comparable elements x and y in a lattice, such that x$\prec$y, the interval [x, y] consists of all the lattice elements z verifying (x$\prec$z $\wedge$ z$\prec$y)

.	p	P	b	B	d	D	f	F	m	M	o	O	e
p	p	pP	p	p	pd	p	pd	p	p	pd	p	pd	p
P	pP	P	dP	P	dP	P	P	P	dP	P	dP	P	P
b	p	P	b	bB	d	pD	d	po	p	M	po	dO	b
B	pD	P	bB	B	dO	D	O	D	oD	M	oD	O	B
d	p	P	d	dP	d	pP	d	pd	p	P	pd	dP	d
D	pD	DP	oD	D	oO	D	DO	D	oD	DO	oD	DO	D
f	p	P	d	OP	d	DP	f	Ff	m	P	od	OP	f
F	p	DP	o	D	od	D	Ff	F	m	DO	o	DO	F
m	p	DP	m	m	od	p	od	p	p	Ff	p	od	m
M	pD	P	dO	P	dO	P	M	M	bB	P	dO	P	M
o	p	DP	o	oD	od	pD	od	po	p	DO	po	oO	o
O	pD	P	dO	OP	dO	DP	O	DO	oD	P	oO	OP	O
e	p	P	b	B	d	D	f	F	m	M	o	O	e

Table 1: Allen's transitivity table

3 Generalized Interval Calculus

Allen's proposal is subject to various extensions. MB. Vilain's contribution [Vila82] principally consists in extending the scope of the study to the mutual relation between points on the one side and convex intervals on the other. It is not possible to consider patterns mixing points and intervals without inviting inescapable ambiguity. MB. Villain's extension preserves or adapts all of Allen's algebraic and relational properties.

3.1 Non convex intervals vs linear point-pattern

Ligozat deals with the mutual relation between sets of points provided with a linear order relation. The dissymmetry of the approach clearly appears: the second set of points splits the linear space into a series of adjacent zones and constitutes a reference system; then, all members in the first set of points are labelled by the rank of the zone in which they lay. This is a straightforward extension to the labelling technique described in 2.4.2.
This switching from a geometric viewpoint to a qualitative one, gives a more abstract knowledge about the situation, and skips the numerical value of coordinates. Similarly, switching from point patterns to interval patterns permits a gain in abstraction.
There is a strict trivial correspondence between series of points and non convex intervals, provided the size of the series of points is even.

In the remainder of the paper, the set of non convex intervals of size p will be denoted by NCI(p). NCI denotes the union of NCI(p) over all p values. Any element in NCI is then a set of disjoint convex intervals (in Allen's sense):

$$\text{NCI} = \{\, \Xi \in \text{Pow}(\mathcal{I}[\mathcal{D}, \leq]) \;\mid\; (\forall X, Y \in \Xi \mid \text{not}(X\underline{e}Y) \bullet X\underline{p}Y \vee X\underline{P}Y\,)\,\}$$

3.2 Ligozat's extensions to Allen's Interval Algebra

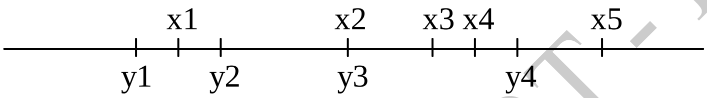
G. Ligozat builds a general theory that extends Allen's and Vilains's results to any kind of linear point-pattern. We first give some definitions, and then show how Allen's results are extended in a natural manner before eventually addressing the special case of non convex intervals.

3.2.1 Basic Principle and Notation

Let X (resp. Y) be a point pattern of size m (resp. size n) such that: X = (x1, …, xm) and Y = (y1, …, yn)). Elements in Y are mapped to the beginning sequence of odd numbers (1, 3, …, 2n-1), thus resulting in defining 2n+1 adjacent zones, merely labelled from 0 to 2n. X is then represented by the series of the labels indicating the area of its elements, according to the following rule:

$$
\begin{array}{llll}
(\forall\ i : 1..\ m \ \bullet & \text{label(xi)} = 2k-1 & \Leftrightarrow & \text{xi} = \text{yk} \quad \wedge \\
 & \text{label(xi)} = 0 & \Leftrightarrow & \text{xi} \leq \text{y1} \quad \wedge \\
 & \text{label(xi)} = 2n & \Leftrightarrow & \text{xi} \geq \text{yn} \quad \wedge \\
 & \text{label(xi)} = 2k & \Leftrightarrow & \text{xi} \in\ ]\ \text{yk/2},\ 1+\text{yk/2}\ [\\
)
\end{array}
$$

In this context, the representation of X is a non decreasing sequence π of m numbers, chosen between 0 and 2n, with no duplicates allowed for odd numbers. π uniquely identifies one configuration, and any configuration has one π code.
As an example, the configuration below, where m = 5 and n = 4 will be coded (2,5,6,6,8)



In the following, the set of possible relations between X and Y only depends on m and n, and will then be denoted as: Π(m,n). For instance, in the above example, π = (2,5,6,6,8) is a member of Π(5,4).
Accordingly, Π will represent the set of all possible Π(m,n), whatever m and n.
From now on, we shall admit to use the π code for indicating a special relation between two non convex intervals, in the same manner we did with convex ones. i.e.: writing X$\underline{\pi}$Y for two intervals in NCI, instead of X$\underline{R}$Y (assuming R is actually coded by π).
As π is a sequence of natural numbers, π(i) will represent the i^{th} element in π.

3.2.2 Properties

Lattice structure

Ligozat's generalization preserves most of the combinatorial properties previously observed in Allen's theory. More precisely, for any integers p and q, Π(p,q) is provided with a distributive lattice structure. The underlying order of the lattice is but similar to that in the simple Allen's case, and results from the product order on the natural numbers in the configuration code.

Inversion *vs* transposition

Due to the dissymmetry evoked above when p ≠ q, inversion is not appropriate for treating of non convex interval mutual relations. In contrast, transposition is convenient for switching the roles of the two intervals involved in a binary relation.
The definition of transposition is such that:

$$
\begin{array}{l}
\forall m,n\text{: NAT1; } R \in \Pi(m,n) \bullet \\
\qquad \exists R^{t} \in \Pi(n,m) \ \bullet\ (\forall X \in NCI(m); Y \in NCI(n) \bullet X\ R\ Y \ \Leftrightarrow Y\ R^{t}\ X)
\end{array}
$$

As regards Ligozat's code, the transposed of π is denoted as π^{t}.

G. Ligozat [Ligo91] gives a precise formula for computing π^t from π. It clearly appears from the definition that all elements in $\Pi(p,q)$ have one unique transposed, and that transposition is an involution.

<u>Composition</u>

The composition $\pi1 ; \pi2$ of two elements respectively members of $\Pi(m1,n1)$ and $\Pi(m2,n2)$ is sound as soon as $n1 = m2$. The result may be indeterminist (when only qualitative information is specified) and provides a set of elements in $\Pi(m1,n2)$ which necessarily constitute an interval in the corresponding lattice.
Ligozat points out fine properties namely that transposition is an order reversing bijection and that transposing the result of the composition of two given elements is equivalent to composing their transposed.

<u>Set Theoretics</u>

With respect to set theoretics, it may appear tempting to develop a scheme similar to Allen's, Unfortunately, the size of Π exponentially increases with p and q. Besides, not many atomic relations are of practical interest. In fact, cases where the mutual relation between convex intervals components is too heterogeneous along the non convex parent interval, prove to be intricate and suffer a lack of abstraction. In such situations, it is valuable to split the set of non convex intervals so as to refer – for each part - to simpler relations.

Specifying a set of meaningful basic relation types while keeping at a sufficient generality level and studying the useful properties of the resulting framework is the goal of the next Section.

4 Selected Synthetic Relations

In this section, we aim at specifying a set of simple and significant body of concepts by typing the relations between non convex intervals. Here, "simple" means that such concepts correspond to common situations frequently encountered by users. As a non restrictive guideline, one current scenario we address, concerns the constitution of a knowledge base for recording and querying data about event occurrences and resource availability.

Ligozat's system with its composition calculus provides a very convenient framework for indexing such information. There remains to achieve the selection of reduced tractable subsets in Π and the constitution of an operational basis for expressing and managing the end user knowledge and processes. Of course, the simple case of two convex intervals addressed by Allen clearly encompasses the concepts of prior interest that should necessarily be considered and adapted for dealing with non convex intervals.

In the next subsection, we first advocate for a founding principle we shall evenly refer to when extending Allen's relations to non convex intervals.

Then, each extended relation in our proposal is specified in two equivalent ways:

- With a set of constraints imposed upon the pairs of convex components in each non convex interval involved in the extended relation. These constraints are predicates which only refer to genuine Allen's relations.
- By directly constraining the Ligozat's code of both non convex intervals in hand.

Then we discuss the properties of these extensions, treating of composition and of other aspects, such as filtering non convex intervals. This last point permits to aid breaking complex relations into simpler ones as evoked above.

4.1 Basic Principle for Extending Allen's Relation to Non Convex Intervals

In the following, we are guided both by the will to take advantage of end users' practice, and by the optimal properties of Allen's genuine relations. Our approach aims at providing a consistent semantics to Allen's relations within the scope of non convex intervals. Hence what we propose can be seen as an extension of Allen's terminology.
A naïve attitude would be to build the extension of ALLEN by imposing a bijection between the components of the two considered non convex intervals, and then to assert that all resulting mapped convex intervals uniformly satisfy one given Allen relation. This is far too much restrictive, and would apply in no case but trivial (especially setting $p = q$).

Our proposal consists in a relaxation of the naïve constraints above, entailing, in the general case, a mere one to one mapping from X to Y instead of a bijection. More precisely, defining an extension R* to Allen's genuine relation R, we give prominence to the first non convex interval (cf.: dissymmetry actually introduced by Ligozat's coding), and ensure that for each of its convex members (e.g.: i1), there will be at least one convex component in the second non convex interval (e.g.: i2) such that $i1\underline{R}i2$ is satisfied. Our fundamental interpretation is that R is uniformly true among all the convex components of the first non convex interval. This strict frame will be discussed in the next subsection.
This kind of knowledge is actually sufficient in many use cases, for instance when querying a knowledge base searching for the activities one user can attend. The first pattern indicates the user availability, and the second non convex interval accounts for the (repeated) activity access periods. No matter if there moreover exists some access periods which do not match the user's pattern.

In any other case, additional constraints may be specified, especially by switching the roles of the two non convex intervals in hand, hence allowing to express more restrictive properties, and especially to rephrase the naïve case evoked above, in which both the first and the second non convex intervals are strictly constrained.

4.2 ALLEN*: A Set of Extended Basic Relations Types between Non Convex Intervals

This section lists the set of extensions we propose, and then enounces their main properties. Beforehand, we need to specify the context of our proposal in a more formal way.
Any genuine Allen's relations applies to $\mathcal{I}[\mathcal{D}, \leq] \times \mathcal{I}[\mathcal{D}, \leq]$ and returns a value in ALLEN.
The set of non convex intervals (NCI) may be expressed as:

$$\text{NCI} \in \text{POW}(\mathcal{I}[\mathcal{D}, \leq]) \;\wedge$$
$$\forall X \in \text{NCI};\; i1, i2 \in \mathcal{I}[\mathcal{D}, \leq] \mid i1 \in X \wedge i2 \in X \;\wedge i1 \neq i2 \;\bullet\; i1\underline{p}\, i2 \;\vee\; i1\underline{P}i2$$

Each element in ALLEN is an abstraction for a series of paired convex interval instances, all instances sharing the same qualitative relationship. Building ALLEN*, we intend to go further in abstraction. It would not be sufficient to mimic this definition, asserting: "each of ALLEN* relation types is an abstraction for a series of paired non convex interval instances, all instances sharing the *same qualitative extended relationship*". In fact, the appropriate abstraction level is not *one* extended relation, but *a set* of such extended relations.

So, ALLEN* elements are (consistent) sets of extended relations:

$$\text{ALLEN}^* \subseteq \text{POW}(\Pi)$$

In the following, it should be kept in mind that when specifying X R*Y, we strictly mean that there exists one element R in Π such that X RY is verified. and R is of type R*.
ALLEN* does not fully encompass Π. Many extended relations in Π match no type in ALLEN*.
We call "Garbage" the set of relations not being typed by ALLEN*. In fact, the semantics of such elements cannot be simply expressed hence the denomination. For instance, (0,0,4,6,6,9) is Garbage in Π(6,6).

4.2.1 ALLEN* relations: Context and general remarks

This subsection successively treats of the extensions we specify when applying the basic principle above to each element in ALLEN. The generalization process leads to investigating the mutual relationship between non convex intervals. In order to do so, we need two prior definitions:

Let X (resp. Y) be a non convex interval consisting of p (resp. q) convex interval components, therefore: $X \in NCI(p)$ and $Y \in NCI(q)$
In the following, we assume that *inc_seq(p, q)* represents the set of strictly increasing sequences of p natural numbers greater than one and lower than q:

$$\forall p, q : NAT1 \mid p \leq q \bullet$$
$$\text{inc\_seq}(p,q) = \{ u : \text{seq } 1..q \mid \text{size}(u) = p \wedge \forall i \in \text{dom}(\text{front}(u)) \bullet u(i) < u(i+1) \}$$

These sequences indicate a canonical relation between the convex components of X and those of Y as it has been implicitly evoked in the Basic Principle. The canonical relation often is a mapping which is monotone with respect to the *precedes* Allen's relation. The domain of u refers to the ranks of the convex elements in X while its range refers to the ranks of convex elements in Y.

Because of the mapping underlying the Basic Principle, most of the definitions of the extended relations in ALLEN* entail a cardinality constraint, namely $p \leq q$. One consequence is that whereas Allen identifies six basic relations each accompanied with its homologous converse, plus one symmetrical relation (e.g.: *equals*), ALLEN* actually points out fourteen individuals (i.e./ 14 extended relation types)..
The original *equals* Allen's relation can obviously only hold provided $p = q$. Consequently, when performing the generalization process, *equals* must be broken into its counterparts, namely: *subset** when $p \leq q$ and *Superset** when $p \geq q$; then, the direct translation *equal** can be expressed as the conjunction: *equals** = (*subset** ∧ *Superset**). Similarly, precedes* and Preceded_by* have to be customized so as to fit the extended pattern, hence leading to specifying *interleaves**.
Apart for *subset** and *Superset** and *interleaves** which are new, all other extended relations types are named by appending a star to the original Allen's denomination. So as to help managing this correspondence along the text below, we shall denote by ALLEN*[R1] the extension R1* of R1 and conversely denote ALLEN[R1*] the genuine basis R1 extended by R1*. This viewpoint is but lexical.

4.2.2 ALLEN* extended relations: Definitions

The list of extended relations is given below. According to the lattice structure attached to Ligozat's representations, it is possible - for any given p and q - to point out the minimal and maximal elements for each definition. These extremal elements are systematically specified as well as the type of canonical relation induced by the extended relation in hand.

Definition_1: begins* /code: b* *(precondition : $p \leq q$)*

$X \text{ begins* } Y \Leftrightarrow \forall x \in X \bullet \exists y \in Y \bullet x \text{ begins } y$

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X \text{ begins* } Y \wedge X\ \pi\ Y \bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 4s(i/2) - 2 \wedge \pi(i-1) = \pi(i) -1)$

minimal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 2i-2 \wedge \pi(i-1) = \pi(i) -1$

maximal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 4q - 4p +2i - 2 \wedge \pi(i-1) = \pi(i) -1$

canonical relation: one to one total mapping from X to Y

Definition_1a: Begun_by* /code: B* *(precondition : $p \leq q$)*

$X \text{ Begun\_by* } Y \Leftrightarrow \forall x \in X \bullet \exists y \in Y \bullet y \text{ Begun\_by } x$

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X \text{ Begun\_by* } Y \wedge X\ \pi\ Y \bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i-1) = 4\,s(i/2) - 3 \wedge \pi(i) \geq \pi(i-1) +3$

minimal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 2i \wedge \pi(i-1) = \pi(i)-3$

maximal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 4q - 4p + 2i \wedge \pi(i-1) = \pi(i)-3$

canonical relation: one to one mapping from X to Y

Definition_2: finishes* /code: f* *(precondition : $p \leq q$)*

$X \text{ finishes* } Y \Leftrightarrow \forall x \in X \bullet \exists y \in Y \bullet x \text{ finishes } y$

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X\ \pi\ Y \wedge X \text{ finishes* } Y \bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 4s(i/2) - 1 \wedge \pi(i-1) = \pi(i) -1)$

minimal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 2i -1 \wedge \pi(i-1) = \pi(i)-1$

maximal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 4q - 4p + 2i-1 \wedge \pi(i-1) = \pi(i)-1$

canonical relation: one to one mapping from X to Y

Definition_2a: Finished_by* /code: F* *(precondition : $p \leq q$)*

$X \text{ Finished\_by* } Y \Leftrightarrow \forall x \in X \bullet \exists y \in Y \bullet x \text{ Finished\_by } y$

Equivalent Ligozat's style constraint

$\forall\pi\in\Pi$ | X $\underline{\pi}$ Y $\wedge$ X Finished_by* Y $\bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 4s(i/2) - 1 \wedge \pi(i-1) \leq \pi(i) - 3$

minimal element

$\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 2i-1 \wedge \pi(i-1) = \pi(i)-3$

maximal element

$\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 4q-4p + 2i - 1 \wedge \pi(i-1) = \pi(i)-3$

canonical relation: one to one mapping from X to Y

Definition_3: meets* /code: m* *(precondition : p≤q)*

X meets* Y $\Leftrightarrow$ $\forall x\in X \bullet \exists y\in Y \bullet$ x meets y

Equivalent Ligozat's style constraint

$\forall\pi\in\Pi$ | X meets* Y $\wedge$ X $\underline{\pi}$ Y $\bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 4\, s(i/2) - 3\ )$

minimal element

$\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 2i-3 \wedge \pi(i-1) = max(0\ ;\ \pi(i) -3)$

maximal element

$\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 4q - 4p -3 + 2i \wedge \pi(i-1) = \pi(i-1) -1$

canonical relation: one to one mapping from X to Y

Definition_3a: Met_by* *(precondition : p≤q)*

X Met_by* Y $\Leftrightarrow$ $\forall x\in X \bullet \exists y\in Y \bullet$ x Met_by y

Equivalent Ligozat's style constraint

$\forall\pi\in\Pi$ | X Met_by* Y $\wedge$ X $\underline{\pi}$ Y $\bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i-1) = 4\, s(i/2) -1\ )$

minimal element

$\forall i\in 1..2p \mid ODD(i) \bullet \pi(i) = 2i+1 \wedge \pi(i+1) = min(4q\ ;\ \pi(i)+3)$

maximal element

$\forall i\in 1..2p \mid ODD(i) \bullet \pi(i) = 4q - 4p +2i+1 \wedge \pi(i+1) = min(4q\ ;\ \pi(i)+3)$

canonical relation: one to one mapping from X to Y

Definition_4 : overlaps* /code: o* *(precondition : p≤q)*

X overlaps* Y $\Leftrightarrow$ $\forall x\in X \bullet \exists y\in Y \bullet$ x overlaps y

Equivalent Ligozat's style constraint

$\forall\pi\in\Pi$ | X $\underline{\pi}$ Y $\wedge$ X overlaps* Y $\bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 4\, s(i/2) - 2 \wedge \pi(i-1) \leq \pi(i) - 2$

minimal element

$\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 2i - 2 \wedge \pi(i-1) = max(0\ ;\ \pi(i)-4)$

maximal element

$\forall i\in 1..2p \mid EVEN(i) \bullet \pi(i) = 4q - 4p +2i-2 \wedge \pi(i-1) = \pi(i)-2$

canonical relation: one to one mapping from X to Y

Definition_4a: Overlapped_by* /code: O* *(precondition : p≤q)*

X Overlapped_by* Y $\Leftrightarrow$ $\forall x\in X \bullet \exists y\in Y \bullet$ x Overlapped_by y

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X \; \underline{\pi} \; Y \;\; \wedge \;\; X$ Overlapped_by* $Y \;\; \bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i \in 1..2p \mid EVEN(i) \bullet \; \pi(i-1) = 4\,s(i/2) - 2 \;\wedge\; \pi(i) \geq \pi(i-1) + 2$

minimal element

$\forall i \in 1..2p \mid ODD(i) \bullet \; \pi(i) = 4i-2 \;\wedge\; \pi(i+1) = \pi(i)+2$

maximal element

$\forall i \in 1..2p \mid ODD(i) \bullet \pi(i) = 4q - 4p + 2i \;\wedge\; \pi(i+1) = \pi(i)+2$

canonical relation: one to one mapping from X to Y

Definition_5 : during* /code: d* *(p≤q)*

X during* $Y \Leftrightarrow \forall x \in X \bullet \; \exists y \in Y \bullet \; x$ during $y \;\wedge$

$(\forall x1, x2 \in X \,;\, y \in Y \mid x1$ during $y \wedge x2$ during $y \bullet x1 = x2)$

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X \; \underline{\pi} \; Y \;\; \wedge \;\; X$ during* $Y \;\; \bullet$

$\exists s \in inc\_seq(p,q) \bullet (\forall i \in 1..2p \mid EVEN(i) \bullet \; \pi(i) = 4\,s(i/2) - 2 \;\wedge\; \pi(i+1) = \pi(i))$

minimal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 2(i-1) \;\wedge\; \pi(i+1) = \pi(i)$

maximal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \pi(i) = 4q - 4p + 2i - 1 \;\wedge\; \pi(i+1) = \pi(i)$

canonical relation: one to one mapping from X to Y

In case of *during**, an additional constraint has been set, in order to avoid the degenerated configuration when all convex components of X would occur during a same convex component of Y. The result actually is a one ton one mapping from X to Y, what – in contrast with the degenerated case – does make sense when dealing in practice, with repeated occurrences of events.

Definition_5a: Contains* /code: D* *(precondition : p≤q)*

X Contains* $Y \;\Leftrightarrow \forall x \in X \bullet \; \exists y \in Y \bullet \; x$ Contains y

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X \; \underline{\pi} \; Y \;\; \wedge \;\; X$ Contains* $Y \;\; \bullet$

$(\forall i \in 1..2p \mid ODD(i) \bullet ($

$(\pi(i) \equiv 0 \pmod 4 \wedge \; \pi(i+1) \geq \pi(i) + 4) \;\vee$

$(\pi(i) \equiv 1 \pmod 4 \wedge \; \pi(i+1) \geq \pi(i) + 7) \;\vee$

$(\pi(i) \equiv 2 \pmod 4 \wedge \; \pi(i+1) \geq \pi(i) + 6) \;\vee$

$(\pi(i) \equiv 3 \pmod 4 \wedge \; \pi(i+1) \geq \pi(i) + 5) \;\vee$

)

)

minimal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \; \pi(i) = 2i \;\wedge\; \pi(i-1) = \pi(i) - 4$

maximal element

$\forall i \in 1..2p \mid EVEN(i) \bullet \; \pi(i) = 4q - 4p + 2i \;\wedge\; \pi(i-1) = \pi(i) - 4$

canonical relation: one to many left-total relation from X to Y

Due to the more restrictive constraint set upon during* one can check that stating (X Contains* Y $\wedge$ Y during* X) requires p = q, and implies there exists a strict one to one correspondence between X and Y.

The next two definitions result from extending the genuine Allen's *equals* relation to cases when p differs from q. They render a set theoretic viewpoint: X subset* Y means that all components of X are also (*equals*) components of Y. In contrast, X equals* *Y* means a strict identity between X and Y components, hence a unique instance in the relation type (namely Y itself) which plainly is both a minimal and maximal element. *Superset** and *subset** are mutually exclusive unless (p = q), in which case they both coincide with *equals**.

Definition_6a: subset* /code: s* *(precondition : $p \leq q$)*

$$X \text{ subset* } Y \Leftrightarrow \forall x \in X \bullet \exists y \in Y \bullet x \text{ equals } y$$

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X\ \pi\ Y \wedge X \text{ subset* } Y \bullet$

$$\exists s \in \text{inc\_seq}(p,q) \bullet (\forall i \in 1..2p \mid \text{EVEN}(i) \bullet \pi(i) = 4\,s(i/2)-1 \wedge \pi(i-1) = \pi(i) - 2)$$

minimal element

$$\forall i \in 1..2p \mid \text{EVEN}(i) \bullet \pi(i) = 2i-1 \wedge \pi(i-1) = \pi(i)-2$$

maximal element

$$\forall i \in 1..2p \mid \text{EVEN}(i) \bullet \pi(i) = 4q - 4q + 2i - 1 \wedge \pi(i-1) = \pi(i)-2$$

canonical relation: one to one mapping from X to Y

Definition_6b: Superset* /code: S* *(precondition : $q \leq p$)*

$$X \text{ Superset* } Y \Leftrightarrow Y \text{ subset* } X$$

minimal element

$(\forall i \in 1..(2q-2p) \bullet \pi(i) = 0) \wedge (\forall i \in (2p-2q)..2q \mid \text{ODD}(i) \bullet \pi(i) = 2i-1 \wedge \pi(i+1) = \pi(i)+2))$

maximal element

$(\forall i \in 2q..2p \bullet \pi(i) = 4q) \wedge (\forall i \in 1..2q \mid \text{ODD}(i) \bullet \pi(i) = 2i-1 \wedge \pi(i+1) = \pi(i)+2))$

canonical relation: one to one partial mapping from X onto Y

Corollary_6: equals* /code: e* *(precondition : $p=q$)*

$$X \text{ equals* } Y \Leftrightarrow X \text{ subset\_in* } Y \wedge X \text{ Superset* } Y$$

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X\ \pi\ Y \wedge X \text{ equals* } Y \bullet (\forall i \in 1..2q \bullet \pi(i) = 2i-1)$

minimal element: $(\forall i \in 1..2q \bullet \pi(i) = 2i-1)$

maximal element: $(\forall i \in 1..2q \bullet \pi(i) = 2i-1)$

canonical relation: one to one correspondence (identity) from X on to Y

There are actually no means to extend precedes and Preceded_by Allens's relations in a satisfactory way, i.e. a way that preserves the repeated scheme of the occurrences of the convex components in non convex intervals. Therefore, we adopt first a rough definition below, stating X *precedes** Y *iff* all components in X do precedes all elements in Y. *Preceded_by** is defined accordingly.

Besides, we define a novel relation type, namely *interleaves** which – in contrast - conveniently accounts for a repeated precedence pattern, well fitted for treating of non convex intervals.

Definition_7: precedes* /code: p* *(precondition: True)*

$$X \text{ precedes* } Y \Leftrightarrow \forall x \in X; y \in Y \bullet x \text{ precedes } y$$

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X\ \underline{\pi}\ Y \ \wedge\ X$ precedes* $Y \ \bullet\ (\forall i \in 1..2p \bullet\ \pi(i) = 0)$

minimal element

$\forall i \in 1..2p \bullet\ \pi(i) = 0$

maximal element

$\forall i \in 1..2p \bullet\ \pi(i) = 0$

canonical relation: many to many total relation from X onto Y (as a matter of fact, the relation corresponds to a complete bipartite graph).

Definition_7a: Preceded_by* /code: P* *(precondition: True)*

X Preceded_by * Y $\Leftrightarrow$ $\forall x \in X;\ y \in Y \bullet$ x Preceded_by y

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X\ \underline{\pi}\ Y \ \wedge\ X$ Preceded_by* $Y \ \bullet\ (\forall i \in 1..2p \bullet\ \pi(i) = 4q)$

minimal element

$\forall i \in 1..2p \bullet\ \pi(i) = 4q$

maximal element

$\forall i \in 1..2p \bullet\ \pi(i) = 4q$

canonical relation: many to many left-total relation from X onto Y

Definitions 7 and 7a obviously are mutually exclusive.

Definition_8 : interleaves* /code: i* *(precondition : $p > 1 \ \wedge\ p \leq q+1$)*

X interleaves* Y $\Leftrightarrow$

$\forall x1, x2 \in X \mid x1$ precedes $x2 \ \bullet\ (\exists\, y \in Y \bullet\ x1$ precedes $y \ \wedge\ y$ precedes $x2)$

Equivalent Ligozat's style constraint

$\forall \pi \in \Pi \mid X\ \underline{\pi}\ Y \ \wedge\ X$ interleaves* $Y \ \bullet$

$$
\begin{aligned}
(\forall i \in 1..2p-2 \mid \mathrm{EVEN}(i) \bullet (\ & (\pi(i) \equiv 0 \pmod 4 \wedge \pi(i+1) \geq \pi(i) + 4) \ \vee \\
& (\pi(i) \equiv 1 \pmod 4 \wedge \pi(i+1) \geq \pi(i) + 7) \ \vee \\
& (\pi(i) \equiv 2 \pmod 4 \wedge \pi(i+1) \geq \pi(i) + 6) \ \vee \\
& (\pi(i) \equiv 3 \pmod 4 \wedge \pi(i+1) \geq \pi(i) + 5) \ \vee \\
)& \\
)&
\end{aligned}
$$

minimal element

$\forall i \in 1..2p \mid \mathrm{EVEN}(i) \bullet\ \pi(i) = 2i - 4 \ \wedge\ \pi(i-1) = \pi(i)$

maximal element

$\forall i \in 1..2p \mid \mathrm{EVEN}(i) \bullet\ \pi(i) = 4q - 4p + 2i \ \wedge\ \pi(i-1) = \pi(i)$

canonical relation: one to many relation from X to Y (except for the first or the last element in X which may have no mapped value).

This latter case is somewhat different from the previous ones since three convex components instead of two are involved in the predicate defining *interleaves** (x1 and x2 in X, and y in Y). As a matter of fact, the definition can be simplified; there is no real need to constrain all pairs of convex components in X as written above. It is enough to ensure that there is at least one element y in Y located between any element x1 in X and its direct successor x2.

So any non extremal element x in X can exactly be mapped to two elements in Y: the last element y1 verifying: (y1 precedes x), and the first element y2 such that: (x precedes y2). Naturally, the first (resp. last) convex component in X may lack a left (resp. right) mapping.

A strict interleaving, i.e.: an actual alternate sequence of elements in X and Y, can be specified by: (X interleaves* Y $\wedge$ Y interleaves* X). In this case, p and q are such that (abs(p-q) ≤ 1). The *interleaves** relation brings no information about which interval starts and terminates the combined series; hence the possible use of *precedes** and/or *Preceded_by** to tell more about the interleaving pattern.

4.2.3 Extended relation calculus

The counterpart for the semantic gain when defining synthetic extended relations, is a kind of indeterminism. More precisely, there are cases when two different extended relations can be verified at a time.

Let's for instance consider X = (3,5,7,13) in Π(4,8): both X Met_by* Y and X meets* Y relations are satisfied. In such a situation, any property of one or other of the two relations does apply to X and Y. Table2 gives an overview of the possible conjunction of two different extended relations. This depends on the relative values of p and q.

Keeping this point in mind, we shall successively discuss the composition of two extended relations and the filtering of components *via* the canonical relation evoked above.

Dealing with composition needs to differentiate between two kinds of extended relations, those which directly derive from genuine Allen's theory and the others. Each case will be treated successively.

Concomitance	p*	P*	b*	B*	d*	D*	f*	F*	m*	M*	o*	O*	s*	i*	e*
p*		F	F	F	F	F	F	F	F	F	F	F	F	F	F
P*	F		F	F	F	F	F	F	F	F	F	F	F	F	F
b*	F	F		F	F	F	F	F	F	F	F	F	F	T2	F
B*	F	F	F		F	T3	F	T3	T3	F	T3	F	F	T2	F
d*	F	F	F	F		F	F	F	F	F	F	F	F	T2	F
D*	F	F	F	T3	F		F	T3	T3	T3	T3	T3	F	T2	F
f*	F	F	F	F	F	F		F	F	F	F	F	F	T2	F
F*	F	F	F	T3	F	T3	F		F	T3	F	T3	F	T2	F
m*	F	F	F	T3	F	T3	F	F		T1	F	T1	F	T2	F
M*	F	F	F	F	F	T3	F	T3	T1		T1	F	F	T2	F
o*	F	F	F	T3	F	T3	F	F	F	T1		T1	F	T2	F
O*	F	F	F	F	F	T3	F	T3	T1	F	T1		F	T2	F
s*	F	F	F	F	F	F	F	F	F	F	F	F		T2	F
i*	F	F	T2	T2	T2	T2	T2	T2	T2	T2	T2	T2	T2		F
e*	F	F	F	F	F	F	F	F	F	F	F	F	F	F	

(F = False), (T1=True) ⇒ q > p, (T2=True) ⇒ q ≥ 2p-1, (T3=True) ⇒ q ≥ 2p

Table 2: Predicate = "Both extended relations may coexist for at least one pair of non convex intervals"

Another useful preliminary result concerns the mutual relationship between extremal elements for each extended relation as defined in Sub-section 4.2.1.

For any given p less than q (or equal), these elements inherit the lattice structure on Π(p,q). Figure 2 (case of p<q) and Figure 3 (case p=q) give the Hasse diagrams for the sub-lattices restricted to such extremal elements. Elements are coded with a sign (namely – or +) followed by a letter.

The letter is the initial of the extended relation (star is omitted) and the sign is *minus* for a minimal element and *plus* for a maximal one. For instance, -P is a shortcut for "minimal element of *Preceded_by**".

In case of p = q, minimal and maximal elements are equal for all relations which induce a one to one canonical mapping. If so, no sign is added to the coding letter.

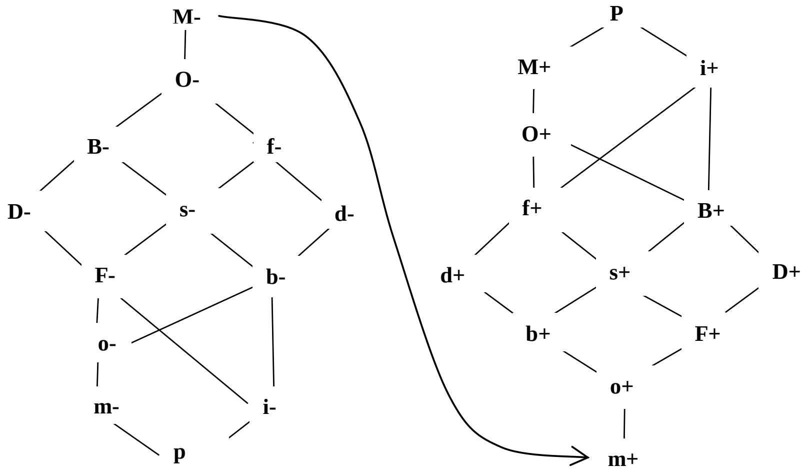


Fig. 2: Hasse diagrams for the lattice structure on extremal elements of extended relations in ALLEN* (Cas: p < q)

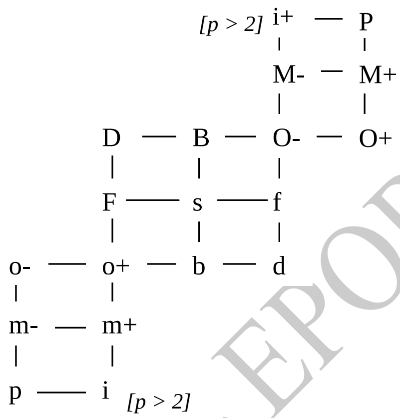


Fig. 3: Hasse diagrams for the lattice structure on extremal elements of extended relations in ALLEN* (Cas: p = q)

4.2.4 Transposition

The denomination of ALLEN* elements records the semantics of genuine Allen's relations and in particular the pairing between one relation and its converse. So, with respect to a mere lexical point of view, in ALLEN*, any element can be paired with its converse according to Allen's sense. In Subsection 3.2.1, we evoked how transposition was a substitute for the converse relationship between extended relations in Π.

In the general case, ALLEN* is not stable for transposition, in the following sense. Let us assume X *R** Y. Then, we infer there exists some R in Π such that X $\underline{R}$ Y. Consequently R^t also belongs to Π, but there is no evidence that R^t is of type $R*^t$, and no evidence either, that R^t belongs to ALLEN*.

For instance: $\pi = (1,2,9,10) \in \Pi(4,6)$ and $\pi \in$ *begins**
then $\pi^t = (1,4,4,4,5,8) \in \Pi(6,4)$ but $\pi^t \notin$ Begun_by*; besides $\pi^t \notin$ ALLEN*

The stability for transposition only applies for the special pair: (precedes*, Preceded_by*) and when p = q. In this latter case, all paired relations prove to be mutual inverses.

In Subsection 4.2.6 which is dedicated to Filtering, further results and properties will be stated about transposition. Beforehand, composition should be investigated.

4.2.5 Composition

Provided some obvious cardinality constraints, ALLEN* extended relations can be composed with one another. The current context is made more complex than when dealing with genuine ALLEN relations, since elements of ALLEN* are *sets* of binary relations instead of individuals. By composing two elements - say R1* ⨾ R2* - in ALLEN*, we mean composing any representative R1 in R1* with any R2 in R2*.

Then, it clearly appears that ALLEN* is not stable by composition. Computing R1 ⨾ R2 can provide either ALLEN* elements, or what we call “Garbage”.

Nonetheless, referring to the Basic Principle which governs ALLEN* definitions and to the general algebraic properties enounced by Ligozat [Ligo91], transitivity tables can be given which indicate all possible results of composition in ALLEN*.

We do not provide the tables themselves but enounce one algorithm for computing them.

<u>Computing the possible results for composition in ALLEN*</u>

Let’s consider two elements R1* and R2* in ALLEN* and R1, R2 in ALLEN such that R1* = ALLEN*[R1] and R2* = ALLEN*[R2].

In the following, we shall assume that: subset* = ALLEN*[equals]. Some special cases such as interleaves*, precedes* and Preceded_by*, which have no direct counterpart in ALLEN will be discussed later on.

Let Γ be the set of possible results for R1* ⨾ R2*; i.e.: for any configuration π in Γ, there are at least two configurations $\pi 1$ in R1* and $\pi 2$ in R2* such that: $\pi 1 \fatsemi \pi 2 = \pi$.

It is clear that any composition result computed with ALLEN’s transitivity tables still applies to ALLEN* calculus then:

$$\text{ALLEN*}[R1 \fatsemi R2] \subseteq \Gamma$$

Besides, additional results can be reached and a superset for them can be computed as follows. Due to the fact that given p and q (cf. Table 2.), ALLEN* relation types are not exclusive, R1* and R2* can be associated with their concomitants, namely $C1 = \{R_{11}^*, \ldots, R_{1k}^*, \ldots, R_{1K}^*\}$ and $C2 = \{R_{21}^*, \ldots, R_{2m}^*, \ldots, R_{2M}^*\}$ respectively.

Then we conclude:

$$\Gamma \subseteq \text{UNION}_{k:1..K;m:1..M}\{\ \text{ALLEN*}[\text{ALLEN}[R_{1k}^*] \fatsemi \text{ALLEN}[R_{2m}^*]]\ \} \qquad \textit{(st\_1)}$$

This strictly means that there are no composition results for R1*⨾ R2* can be found out of RHS of the above statement. Nevertheless, even if concomitants do exist for R1* and R2*, some of the RHS elements can in no way (i.e.: whatever p and q) be obtained after composition, because of intrinsic topological constraints. All other elements can actually be obtained provided some constraints are satisfied such as $(q \geq p)$ and $(r \geq p+q)$ …

These unreachable elements are listed below.

row ; col	b	B	o	O	m	M	f	F	C	s
b				BM		BO				
B				BM		BO				
o	Fm						Fm		Fm	Fm
O	BM						BM		BM	BM
m	Fo	Fo		m		m	Fm		Fm	Fo
M	BM		M		M		BO	BO	BM	BO
f			Fm		Fo					
F			Fm		Fo					
s			Fm	BM	Fo	BO				

Fig. 4: Invalid results for composition of ALLEN* elements

Once these invalid elements withdrawn, the inclusion stated in (st_1) becomes a strict equality which entirely specifies Γ for those elements in ALLEN* which have an ALLEN counterpart

Special cases (elements in ALLEN* with no directe counterpart in ALLEN)

Interleaves* clearly remains out of scope in the previous discussion. As a matter of fact, any composition can result in interleave* provided ($r \geq p+q-1$)

precedes* cannot be obtained except in case precedes* is one of the composed elements. The same property stands for **Preceded_by***.

4.2.6 Filtering

The previous sections treat of relational and logical aspects of non convex intervals. In the present section, we intend to adopt a constructive, operational set theoretic approach. We focus on elements X and Y in NCI as sets of convex components $X=(x_i)_{i=1:p}$ and $Y=(y_j)_{j=1:q}$. According to the definition of ALLEN*, we stressed the so called 'canonical relation' (cf. Subsection 4.2.1) which pointed out the set of paired components (respectively of X and Y) actually being instances of the extended relation in hand. We provide below several operators that permit to select these special paired elements which are the core of the mutual relationship between X and Y, and drop the others. This is what we call 'filtering'.
A set theoretic calculus can be developed on this basis, which proves to allow more efficient reasoning upon non convex intervals. We shall first give the preliminary definitions, and then state some resulting properties

Definition : ALLEN Filters

Given any Allen relation, let us define a filter as follows:

$$\forall R\text{: ALLEN; } X,Y\text{: NCI} \bullet$$
$$R[X,Y] = \{RX,RY\text{: NCI} \mid RX \subseteq X \ \wedge RY \subseteq Y \wedge (\forall x \in RX;\ y \in RY \bullet x\ \underline{R}\ y)\}$$

R[_,_] is the filter attached to Allen's relation R, and R[X,Y] which belongs to NCI x NCI is the result of applying the 'R filter' to X and Y

Definition: Projectors

It is also useful to define two projectors so as to access either filtered elements, namely Pr1.R[_,_] and Pr2.[_,_] which are specified as:

$$\forall R\text{: ALLEN; } X,Y\text{: NCI} \bullet$$
$$Pr1.R[X,Y] = \{x \in X \mid (\exists y\text{: } Y \bullet x\ \underline{R}\ y)\ \} \wedge Pr2.R[X,Y] = \{y \in Y \mid (\exists x\text{: } X \bullet x\ \underline{R}\ y)\ \}$$

Definition : ALLEN* Filters

The definitions above can be extended to ALLEN*. The effect of applying such a filter (R*[_,_]) or such projectors (Pr1.R*[_,_], Pr2.R*[_,_]) is to drop all convex elements not being involved in the canonical relation, and to retain others.

Properties

The filters and the projectors are idempotent.
The following property applies for all R in ALLEN, and for all X and Y in NCI, assuming R* = ALLEN*[R] and R~* = ALLEN*[R~] where R~ denotes the converse of R in ALLEN.

Prop.1): (Pr1.R[X,Y]) R* (Pr2.R[X,Y]) $\wedge$
(Pr2.R[X,Y]) R~* (Pr1.R[X,Y]) $\wedge$
(Pr1.R[X,Y]) R* Y $\wedge$
(Pr2.R[X,Y]) R~* X

Prop.2): X R* Y $\Rightarrow$ X R* (Pr2.R*[X,Y]) $\wedge$ (Pr2.R*[X,Y]) R~* X

4.2.7 Non Convex Interval complement configuration

In contrast with the convex interval case, the complement is worth consideration in the non convex case. We shall first give a precise definition of the complement before pointing out some properties, and indicating how this concept reduces the Garbage in Π.

Definition : Complement configuration of a non convex interval.
Let us consider any π in $\Pi(2p,2q)$. The complement configuration of π is denoted $\neg\pi$ and can be computed in the following way.
π is associated with a sequence of integers. Elements in π are paired so as to indicate the convex components. The beginning of the sequence is marked by “^^” and its end by “$$”.

So, for instance, considering $\Pi(4,6)$ and π =(2, 6, 6, 8) which pertains to O*, the corresponding sequence is ^^2,6 - 6,6 - 6,8$$
The complement of π is $\neg\pi$ = (0,2,6,6,8,12) it belongs to $\Pi(6,6)$
Let’s consider two more examples:
π1 = (0,4,4,8) then $\neg\pi$1 = (4,4,8,12) in $\Pi(4,6)$
π2 = (4,4,8,12) then $\neg\pi$ = (0,4,4,8) in $\Pi(4,6)$

The rules for computing $\neg\pi$ out of π in $\Pi(2p,2q)$ are given below.
We systematically consider every two adjacent pairs in the sequence, and both ^^ and $$ are considered as (virtual) pairs.

R1:	x,y – z,t	$\rightarrow$	y,z				
R2:	^^0,y	$\rightarrow$	void	R4:	z,2q$$	$\rightarrow$	void
R3:	^^x,y	$\rightarrow$	0,x	R5:	z,t$$	$\rightarrow$	t,2q

Properties

P1: Provided the rules above, the function $\pi \rightarrow \neg\pi$ is an involution (i.e.: $\neg\neg\pi = \pi$)

P2: If π belongs to $\Pi(2p,2q)$, then $\neg\,\pi$ belongs to $\Pi(2p',2q)$ with $p' \in \{p-1, p, 2p+1\}$

According the type of π, that of $\neg\pi$ may be inferred. More precisely we can assert the following. All results must be supplemented with their possible concomitants.

$\pi \in p^* \Rightarrow \neg\pi \in$ Garbage
$\pi \in P^* \Rightarrow \neg\pi \in$ Garbage
$\pi \in b^* \Rightarrow \neg\pi \in m^*$
$\pi \in B^* \Rightarrow \neg\pi \in m^*$
$\pi \in d^* \Rightarrow \neg\pi \in o^*$
$\pi \in D^* \Rightarrow \neg\pi \in i^*$
$\pi \in f^* \Rightarrow \neg\pi \in M^*$
$\pi \in F^* \Rightarrow \neg\pi \in M^*$
$\pi \in m^* \Rightarrow \neg\pi \in B^*$
$\pi \in M^* \Rightarrow \neg\pi \in F^*$
$\pi \in o^* \Rightarrow \neg\pi \in O^*$
$\pi \in O^* \Rightarrow \neg\pi \in o^*$
$\pi \in s^* \Rightarrow \neg\pi \in$ Garbage
$\pi \in i^* \Rightarrow \neg\pi \in$ Garbage
$\pi \in e^* \Rightarrow \neg\pi \in$ Garbage

As complements of some ALLEN* elements provide garbage, and thanks to the involution property, it clearly appears that some elements in the garbage have their complements in ALLEN*. In such cases, it is preferable to process the complement instead of the original configuration.

5 Industrial use cases

We applied our proposal to industrial use cases on the occasion of the RMM2 ANR project[2]. Our academic partners (MoDyCo lab.[3]) are involved in Natural Language Processing research and achieved the initial part of the work when analyzing plain text press dispatches issued from our industrial partners (AFP agency[4]) to the benefit of the final user (RelaxNews[5]).

Modelling temporal events

The corpus of dispatches was dedicated to culture and leisure event information management (recording, processing/querying, publication, visualization…). We were in charge of modelling the events intended to be eventually recorded in a knowledge base. We focused on time issues, and had to design special metamodels for structured events with their body of temporal information. Temporal information refers to date-time for events occurrences or for access periods for various resources, e.g.: performances, exhibitions, registration, accommodation… Our role was first to complement, enrich and check the consistency of the output provided by MoDyCo, and then to specify Object Models and design translators in order to leverage the interoperability with visualization frameworks. The whole proposal applied Model Driven Engineering techniques (MDE) most convenient for managing complexity and automating software code generation.

Enrichment, completion and verification were made online when incrementally populating the system, in interaction with a human business practitioner (e.g.: journalist). Our framework exploited the current knowledge base so as to suggest relevant default data, and check the final consistency of the proposed input.

2 RMM2: Projet Relaxmultimedias 2 : http://www.rmm2.org
3 MoDyCo : http://www.modyco.fr/
4 AFP : http://www.afp.com
5 RelaxNews: http://www.relaxnews.com/

Checking the consistency of temporal data clearly relies on such topological knowledge such as that addressed by Allen. Access periods and event occurrences evenly are periodical or at least repeated, hence the need of ALLEN* extensions for taking them in consideration.

Modelling Calendars

A fundamental piece of knowledge about temporal data is embedded in calendars. Our event model, accompanied with ALLEN* definitions allows us to present with a consistent and nice way for modelling calendars as sets of periodic events mutually connected through ALLEN* relations.

Other practical examples have been studied namely within the scope of seashell digging administrative regulation.

6 Conclusion

Driven by industrial requirements and goals, we have been led to provide a practical extension of the genuine Allen relations. Conversely, we based our proposal on a pragmatic restriction of the general theoretical work of Ligozat. Among the extremely large set of possible relations between non convex intervals, we retained special types which represent most of Allen's semantics. Hence, our proposal properly fits the common useful knowledge while keeping to a rigorous theoretical basis.

One relation which is of most common use in practice has been specified (e.g.: interleaves*) and three have been transformed so as to keep making sense within the non convex interval context (preceeds*, Preceded_by* and contains*). Equals has been split into subset* and Superset* since it proves degeneracy insofar non convex intervals are considered.

The composition calculus between extended relations in ALLEN* has been studied and transitivity tables are computed. The distributive lattice structure stands as a general background for comparing relations to one another.
Filtering performs a kind of projection from ALLEN* space onto ALLEN* one.

Present works exploit an Object metamodel which accounts for the intensional form of temporal expressions *(e.g.: "the service will be available between 11 am to 3 pm on the third tuesday of each month from years 2012 to 2014 except in August"*. Intension of temporal expressions is a concise way for recording temporal information about repeated events, compared to recording a set of numerous concrete date-times. Moreover, the temporal semantics is retained in the intensional specification and lost in the extensional one. Lastly, granularity concerns are easier to master within the intensional conceptual framework.

Queries on the temporal knowledge can be directly answered from the intensional form of temporal relations. In fact, the composition calculus can be applied and the semantics of calendars exploited to aid answering the queries. A set of rules can be specified which constitutes the basis of a temporal reasoning engine. If necessary, well known methods can be provided and automatically translate intensional specifications into extensional ones on demand.
Our proposal can surely be extended itself, and address the case of (semi) infinite time intervals. For now, we suppose that a general bounded time extent is a general frame for our reasoning.

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