On a pertubed anisotropic equation with a critical exponent

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Abstract

We study a perturbed anisotropic equation without using the knowledge of the limiting problem. This provides a different method from that introduced by Brzis and Nirenberg [4]. Our arguments use some tools recently developed in [5, 6].

keywords: anisotropic critical exponent, critical level, compactness, Nehari manifold
MCS: 35J25, 35J60, 35J65

1 Introduction.

In this paper, we give new results on critical anisotropic elliptic equations. More precisely, we study the following problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = \lambda a(x)|u|^{q-2}u + |u|^{p^*-2}u \ \text{ in } \ \mathbb{R}^N \tag{1}$$

where $\lambda \geq 0$ is a parameter and the exponents p_i, p^* satisfy the following conditions : $p_i > 1$, $\displaystyle\sum_{i=1}^{N}\frac{1}{p_i} > 1$, $\max\{p_1,\ p_2,\ ...,\ p_N\} < p^*$ and the critical exponent p^* associated to the main operator of (1) is defined by

$$p^* := \frac{N}{\displaystyle\sum_{i=1}^{N}\frac{1}{p_i} - 1}.$$

The function a is a nontrivial and nonnegative function in $L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$. In the isotropic case $p_i = 2$, $i = 1, \cdots, N$, the existence of minimizing solutions in the special case $\lambda = 0$ was completely solved by Aubin [2] and G. Talenti [13]. Their proofs are based on symmetrization theory. Notice that this theory is not relevent in our context since the radial symmetry of solutions can not hold true because of the anisotropy of the operator.

We make precise here that existence results to Problem (1) need in general some precise knowledge of extremal functions to the limiting problem corresponding to $\lambda = 0$ (see for example [4]). In this work, we will carry out a different method.

The natural functional framework of Problem (1) is the anisotropic Sobolev spaces theory developed by [10, 15, 11, 12, 14]. Then, let $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ be the completion of the space $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm $\|u\| := \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|_{p_i}$.

It is well known that $\left(\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N), \|\cdot\|\right)$ is a reflexive Banach space which is continuously embedded in $L^{p^*}\left(\mathbb{R}^N\right)$.

In this work, we deal with the nonlocal existence, with respect to λ, of nontrivial and nonnegative solutions to Problem (1). More precisely, we will show the existence of a characteristic value of λ, denoted by λ^* such that Problem (1) has nontrivial nonnegative solutions for every $\lambda > \lambda^*$. On the other hand, the notion of critical level associated to (1) will be introduced, notice that this notion was first introduced by T. Aubin [1] for the isotropic Laplace operator.

Consider the Euler-Lagrange functional associated to Problem (1) defined by

$$J_\lambda(u) := \sum_{i=1}^{N} \frac{1}{p_i} \int_{\mathbb{R}^N} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} a(x)|u|^q dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx$$

which is of class $C^1(\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N))$. The space $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ can also be seen as

$$\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) = \left\{ u \in L^{p^*}(\mathbb{R}^N) \ : \ \left|\frac{\partial u}{\partial x_i}\right| \in L^{p_i}(\mathbb{R}^N) \right\}.$$

We introduce

$$\mathcal{D}_+^{1,\vec{p}}(\mathbb{R}^N) = \left\{u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) : u \geq 0\right\}.$$

By solutions of Problem (1) we understand critical points of the functional J_λ. Remark that the functional J_λ is bounded neither above nor below on $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$. Then, to find possible critical points of J_λ, we limit the study to the corresponding Nehari manifold which contains all critical points of J_λ. We recall that the Nehari manifold associated to J_λ, denoted by $\mathcal{N}_{J_\lambda}$, is defined by

$$\mathcal{N}_{J_\lambda} := \left\{\varphi \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\} \ : \ J'_\lambda(\varphi)(\varphi) = 0\right\}.$$

In the sequel, we will set $|u|_{a,q} := \left(\int_{\mathbb{R}^N} a(x)|u|^q dx\right)^{\frac{1}{q}}$, $P_i(u) := \int_{\mathbb{R}^N} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx$, $Q(u) := \int_{\mathbb{R}^N} a(x)|u|^q dx$, $P_*(u) := \int_{\mathbb{R}^N} |u|^{p^*} dx$, $p_- = \min\{p_1,\ p_2,\ ...,\ p_N\}$ and $p_+ = \max\{p_1,\ p_2,\ ...,\ p_N\}$. Also,

$$K(u) = \sum_{i=1}^N P_i(u) - P_*(u), \qquad K_+(u) = \max(K(u), 0),$$

$$\gamma_i = \frac{1}{p_i} - \frac{1}{p^*}, i = 1, \ldots, N, \qquad \widetilde{J}_0(u) = \sum_{i=1}^N \gamma_i P_i(u).$$

2 Preliminary results

We start our preliminary results by this elementary lemma

Lemma 1. *The functional*

$$\begin{array}{ccc} \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \displaystyle\int_{\mathbb{R}^N} a(x)|u|^q dx \end{array}$$

is weakly continuous.

Proof. The proof is standard, it can be found in [3] □

We recall that the Nehari manifold can be characterized more explicitly by

$$\mathcal{N}_{J_\lambda} := \left\{ t\varphi \; ; \; (t,\varphi) \in (\mathbb{R} \setminus \{0\}) \times (\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}) \; : \; \frac{d}{dt} J_\lambda(t\varphi) = 0 \right\}.$$

For this reason, we introduce the modified functional

$$\widetilde{J}_\lambda(t,u) := J_\lambda(tu), \text{ on } \mathbb{R} \times \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N).$$

Since we are interested in positive solutions to Problem (1), we restrict ourselves in what follows to $t > 0$.

Lemma 2. *Let $q \in (p_+, p^*)$. Then for every $\lambda \geq 0$ and $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}$, there is a unique $t(u,\lambda) > 0$ such that $t(u,\lambda)u \in \mathcal{N}_{J_\lambda}$. Moreover, the map $(u,\lambda) \to t(u,\lambda)$ is of class C^1 for the strong topology on $\left(\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}\right) \times \mathbb{R}_+$. Furthermore, $t(\gamma u, \lambda) = \dfrac{1}{\gamma} t(u,\lambda)$, for every $\gamma > 0$, and $t(|u|,\lambda) = t(u,\lambda)$.*

The proof is is based on the following proposition.

Proposition 1. *Consider the function*

$$\begin{array}{rcl} \Phi : \mathbb{R}_+^* =]0,+\infty[& \longrightarrow & \mathbb{R} \\ t & \longmapsto & \displaystyle\sum_{i=1}^{m} a_i t^{\alpha_i} - \sum_{j=1}^{n} b_j t^{\beta_j} \end{array}$$

with $\alpha_i, \; a_i \geq 0, \; 1 \leq i \leq m$ and $b_j \geq 0, \; \beta_j > 0, \; b_1 > 0, \; 1 \leq j \leq n$, and $0 < \alpha_i < \beta_j, \; \forall (i,j)$. Then, there is a unique real number $t_0 > 0$ such that $\Phi(t_0) = 0$, $\Phi'(t_0) < 0$, $\Phi(t) < 0$ for $t > t_0$ and $\Phi(t) > 0$ for $0 < t < t_0$. Moreover, Φ has exactly one global maximum on $\mathbb{R}_+^*$.

Proof. It is clear that Φ has at least one local maximum. Moreover, for every $t > 0$, if $\Phi'(t) = 0$ then one has necessarily $\Phi''(t) < 0$. Indeed, if $\Phi'(t) = 0$ then

$$\sum_{i=1}^{m} \alpha_i a_i t^{\alpha_i} - \sum_{j=1}^{n} \beta_j b_j t^{\beta_j} = 0.$$

But $\Phi''(t)$ has the same sign as

$$\sum_{i=1}^{m} \alpha_i^2 a_i t^{\alpha_i} - \sum_{j=1}^{n} \beta_j^2 b_j t^{\beta_j} < \left(\sum_{i=1}^{m} \alpha_i a_i t^{\alpha_i} - \sum_{j=1}^{n} \beta_j b_j t^{\beta_j} \right) \inf_{1 \leq j \leq n} \beta_j = 0,$$

where the first inequality is due to the fact that $\alpha_i < \beta_j$ for every i and j. This ends the proof. □

Now, we return to the:

Proof of Lemma 2. For given $\lambda \geq 0$ and $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \backslash \{0\}$, we consider the function

$$\Phi(t) = J'_\lambda(tu) \cdot tu = \sum_{i=1}^{N} t^{p_i} P_i(u) - t^{p^*} P_*(u) - \lambda Q(u) t^q.$$

We apply Proposition 1 with $\alpha_i = p_i$, $i = 1, \ldots N = m$, $a_i = P_i(u) \geq 0$, $b_1 = P_*(u) > 0$, $b_2 = \lambda Q(u) \geq 0$, $\beta_1 = p^*$ and $\beta_2 = q$. There is a unique $t(u, \lambda) > 0$, such that $\Phi\big(t(u, \lambda)\big) = 0$. The uniqueness implies

$$t(\gamma u, \lambda) = \frac{1}{\gamma} t(u, \lambda) \quad \forall \gamma > 0,$$

and $t(|u|, \lambda) = t(u, \lambda)$ noticing that $P_i(|u|) = P_i(u)$, $P_*(u) = P_*(|u|)$. Since $\frac{\partial \Phi}{\partial t}\big(t(u, \lambda)\big) \neq 0$, and the mappings $u \to P_i(u)$, $u \to P_*(u)$ are of class C^1 for the strong topology, the implicit function theorem shows that the mapping $(u, \lambda) \to t(u, \lambda)$ is of class C^1. □

Next, we want to study the monotonicity of the map $\lambda \to t(u, \lambda)$ for a fixed u.

Lemma 3. *Let $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \backslash \{0\}$. Then the map $\lambda \in \mathbb{R}_+ \to t(u, \lambda)$ is decreasing if $Q(u) > 0$. If $Q(u) = 0$, then $t(u, \lambda) = t(u, 0)$ for every $\lambda \geq 0$.*

Proof. Let us show that the continuous map $\lambda \to t(u, \lambda)$ is injective for $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \backslash \{0\}$ and $Q(u) > 0$. Let (λ_1, λ_2) be such that $t(u, \lambda_1) = t(u, \lambda_2)$ then

$$\phi\big(t(u, \lambda_1)\big) = \phi\big(t(u, \lambda_2)\big),$$

where

$$\phi(t) = \sum_{i=1}^{N} t^{p_i - q} P_i(u) - t^{p^* - q} P_*(u),\ t > 0.$$

By the definition of $t(u, \lambda)$ given in Lemma 2, one has

$$\phi\big(t(u, \lambda_i)\big) = \lambda_i Q(u),\ \text{for } i = 1,\ 2,$$

thus, since $Q(u) > 0$, one has $\lambda_1 = \lambda_2$. Thus the map $\lambda \to t(u, \lambda)$ is necessarily strictly monotone, but by the definition of $t(u, 0)$, if we set $\Phi(t) := \sum_{i=1}^{N} t^{p_i} P_i(u) - t^{p^*} P_*(u)$, one has for every $\lambda > 0$:

$$\Phi\big(t(u, 0)\big) = 0 < \Phi\big(t(u, \lambda)\big).$$

Applying Proposition 1, we conclude that $t(u, \lambda) < t(u, 0)$. This shows that $\lambda \to t(u, \lambda)$ is decreasing. Finally, if $Q(u) = 0$ then $t(u, \lambda) = t(u, 0)$. □

It follows from Lemma 2 that for every $\lambda \geq 0$, on has more precisely

$$\mathcal{N}_{J_\lambda} = \Big\{ \pm\, t(u, \lambda) u \ :\ u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\} \Big\}.$$

At this stage, for every $\lambda \geq 0$, we introduce

$$\alpha(\lambda) := \inf_{u \in \mathcal{N}_{J_\lambda}} J_\lambda(u) = \inf_{u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}} J_\lambda(t(u, \lambda) u).$$

The lemma 2 implies again that the functional

$$\begin{array}{ccc} \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\} & \longrightarrow & \mathbb{R} \\ u & \longmapsto & J_\lambda(t(u, \lambda) u) \end{array}$$

is 0-homogeneous and is even. Then we get

$$\alpha(\lambda) = \inf_{u \in \mathbb{S}} J_\lambda(t(u, \lambda) u), \tag{2}$$

where

$$\mathbb{S} := \left\{ u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \ :\ \sum_{i=1}^{N} P_i(u)^{\frac{1}{p_i}} = 1 \right\}.$$

Lemma 4. *Let*

$$N_0 = \Big\{ v \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)\backslash\{0\} : P_*(v) = \sum_i P_i(v) \Big\} = \mathcal{N}_{J_0},$$

$$\widetilde{N}_0 = \Big\{ v \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)\backslash\{0\} : \sum_i P_i(v) \leq P_*(v) \Big\}.$$

Then for every $\lambda \geq 0$ and $v \in \widetilde{N}_0$ one has $t(v,\lambda) \leq 1$. Furthermore, we have the equivalence: $t(v,0) = 1$ if only if $v \in N_0$.

Proof. Let $v \in \widetilde{N}_0$, from the definition of $t(v,\lambda)$, one has :

$$\sum_i^N t(v,\lambda)^{p_i} P_i(v) = t(v,\lambda)^{p^*} P_*(v) + t(\lambda,v)^q \lambda Q(v) \geq t(v,\lambda)^{p^*} \sum_{i=1}^N P_i(v).$$

Thus :

$$\sum_i \Big(t(v,\lambda)^{p_i - p^*} - 1 \Big) P_i(v) \geq 0.$$

This inequality implies necessarily that $t(v,\lambda) \leq 1$ since $p_i - p^* < 0$ and $v \not\equiv 0$.

By Proposition 1 applied to $\Phi(t) = \sum_{i=1}^N t^{p_i} P_i(v) - t^{p^*} P_*(v)$, if $v \in N_0$ then $\Phi\big(t(v,0)\big) = 0$ since $\Phi(1) = 0$, thus $t(v,0) = 1$ and conversely. □

Lemma 5. *Let $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)\backslash\{0\}$ with $Q(u) > 0$ and $\lambda(u) = \dfrac{K_+(u)}{Q(u)}$. Then*

$$t\big(u,\lambda(u)\big) = \begin{cases} 1 & if\ u \in N_0 \\ < 1 & if\ u \in \widetilde{N}_0\backslash N_0 \\ 1 & if\ u \notin \widetilde{N}_0 \end{cases}$$

Proof. If $u \in N_0$, according to the above lemma 4, $t(u,0) = 1$ and $\lambda(u) = 0$. If $u \in \widetilde{N}_0\backslash N_0$ thus $\lambda(u) = 0$ and $t(u,0) < 1$ (according to Lemma 4). If $u \notin \widetilde{N}_0$ then $\lambda(u) = \dfrac{1}{Q(u)}\left(\sum_{i=1}^N P_i(u) - P_*(u) \right) > 0$, that is

$$\sum_{i=1}^N P_i(u) - P_*(u) - \lambda(u) Q(u) = 0.$$

Considering $\Phi(t) = \sum_{i=1}^{N} t^{p_i} P_i(u) - t^{p^*} P_*(u) - \lambda(u) t^q Q(u)$. We may apply Proposition 1 to conclude that 1 is the unique zero of Φ : $\Phi(1) = 0 = \Phi\Big(t\big(u, \lambda(u)\big)\Big)$ by definition, thus $t\big(u, \lambda(u)\big) = 1$. $\square$

Corollary 1 (of Lemma 5). *Let $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \backslash \{0\}$. Then, the set $\Big\{\lambda \geq 0 : t(u, \lambda) \leq 1\Big\}$ is an interval of the form $\Big]\lambda_{min}(u); +\infty\Big[$ if $Q(u) > 0$ and is empty or equal to $\mathbb{R}_+$ otherwise.*

Proof. By Lemma 5, the set $\Big\{\lambda \geq 0 : t(u, \lambda) \leq 1\Big\}$ is not empty if $Q(u) > 0$ and the fact that $\lambda \to t(u, \lambda)$ is decreassing implies that the set is an interval, we set $\lambda_{min}(u) = \inf \Big\{\lambda \geq 0 : t(u, \lambda) \leq 1\Big\}$. If $Q(u) = 0$ the set is empty if $t(u, 0) > 1$ and is equal to $\mathbb{R}_+$ if $t(u, 0) \leq 1$. $\square$

The link between J_λ and $\widetilde{J}_0$ is carried out in the following:

Lemma 6. *For every $v \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \backslash \{0\}$ and $\lambda \geq 0$, one has*

$$J_\lambda\big(t(v, \lambda) v\big) = \widetilde{J}_0\big(t(v, \lambda) v\big) - \lambda \left(\frac{1}{q} - \frac{1}{p^*}\right) t(v, \lambda)^q Q(v).$$

Proof. We set temporarily $t = t(v, \lambda)$. Then,

$$J_\lambda\big(t(v, \lambda) v\big) = \sum_{i=1}^{N} \frac{1}{p_i} t^{p_i} P_i(v) - \frac{t^{p^*}}{p^*} P_*(v) - \frac{\lambda}{q} t^q Q(v). \tag{3}$$

By the definition of $t(v, \lambda)$, one has

$$\sum_{i=1}^{N} t^{p_i} P_i(v) = t^{p^*} P_*(v) + \lambda t^q Q(v), \tag{4}$$

we deduce from relations (3) and (4) that

$$\begin{aligned} J_\lambda\big(t(v, \lambda) v\big) &= \sum_{i=1}^{N} \left(\frac{1}{p_i} - \frac{1}{p^*}\right) t^{p_i} P_i(v) - \lambda \left(\frac{1}{q} - \frac{1}{p^*}\right) t^q Q(v) \\ &= \widetilde{J}_0\Big(t(v, \lambda) v\Big) - \lambda \left(\frac{1}{q} - \frac{1}{p^*}\right) t(v, \lambda)^q Q(v). \end{aligned}$$

$\square$

3 Palais-Smale sequences on the Nehari manifold

In what follows, we will write $(PS)_c$ to denote a Palais-Smale sequence of J_λ with the level $c \in \mathbb{R}$.

Lemma 7. *Let $\lambda \geq 0$. Then we have the following assertions:*
(i) *Every minimizing sequence $(u_n) \subset \mathbb{S}$ of (2) satisfies*

$$0 < \liminf_{n\to\infty} t(u_n, \lambda) \leq \limsup_{n\to\infty} t(u_n, \lambda) < +\infty.$$

(ii) *There exists a nonnegative minimizing sequence of (2) denoted by $(u_n) \subset \mathbb{S}$ such that $(t(u_n, \lambda)u_n)$ is a bounded Palais-Smale sequence for J_λ.*

Proof. (i) Let us first show that $t(u_n, \lambda)$ is bounded as $n \to +\infty$. Suppose that there exists a subsequence, still denoted by (u_n) such that

$$\lim_{n\to+\infty} t(u_n, \lambda) = +\infty.$$

On one hand, it is clear that

$$\sum_{i=1}^{N} t(u_n, \lambda)^{p_i} P_i(u_n) = t(u_n, \lambda)^q Q(u_n) + t(u_n, \lambda)^{p^*} P_*(u_n). \tag{5}$$

On the other hand, using the usual Sobolev embedding, there is a positive constant c_1 (which is independent of n) such that

$$Q(u_n)^{\frac{1}{q}} \leq c_1 P_*(u_n)^{\frac{1}{p^*}}.$$

It follows that

$$\sum_{i=1}^{N} t(u_n, \lambda)^{p_i} P_i(u_n) \leq c_2 \left(t(u_n, \lambda) P_*(u_n)^{\frac{1}{p^*}}\right)^q + \left(t(u_n, \lambda) P_*(u_n)^{\frac{1}{p^*}}\right)^{p^*}.$$

Then, one has necessarily

$$\lim_{n\to+\infty} t(u_n, \lambda) P_*(u_n)^{\frac{1}{p^*}} = +\infty$$

and consequently,

$$\begin{aligned} t(u_n,\lambda)^q Q(u_n) &= \mathrm{o}_n\left(t(u_n,\lambda)^{p^*}P_*(u_n)\right), \\ \sum_{i=1}^{N} t(u_n,\lambda)^{p_i}P_i(u_n) &= t(u_n,\lambda)^{p^*}P_*(u_n)(1+\mathrm{o}_n(1)). \end{aligned}$$

Therefore, there exists $c_3 > 0$ independent of n, such that

$$\begin{aligned} \lim_{n\to+\infty} J_\lambda(t(u_n,\lambda)u_n) &= \lim_{n\to+\infty}\sum_{i=1}^{N}\left(\frac{1}{p_i}-\frac{1}{p^*}\right)t(u_n,\lambda)^{p_i}P_i(u_n) \\ &\geq c_3 \lim_{n\to+\infty} t(u_n,\lambda)^{p_-}||u_n|| = +\infty. \end{aligned}$$

This is in contradiction with the fact that $\alpha(\lambda)$ is finite.
Now, suppose that there exists a subsequence, still denoted by (u_n) such that

$$\lim_{n\to+\infty} t(u_n,\lambda) = 0.$$

But, it is clear that

$$\frac{\partial^2 \widetilde{J}_\lambda}{\partial t^2}(t,u_n)\Big|_{t=t(u_n,\lambda)} < 0.$$

Then, combining this fact with (5), we get

$$\sum_{i=1}^{N} p_i t(u_n,\lambda)^{p_i}P_i(u_n) < q t(u_n,\lambda)^q Q(u_n) + p^* t(u_n,\lambda)^{p^*}P_*(u_n).$$

Therefore, there exists $c_4 > 0$ independent of n, such that

$$c_4 t(u_n,\lambda)^{p_+}||u_n|| \leq q t(u_n,\lambda)^q Q(u_n) + p^* t(u_n,\lambda)^{p^*}P_*(u_n).$$

Using the assumptions $p_+ < q$ and $p_+ < p^*$, we obtain the following contradiction

$$1 = ||u_n|| \leq \frac{1}{c_4}\left[q t(u_n,\lambda)^{q-p_+}Q(u_n) + p^* t(u_n,\lambda)^{p^*-p_+}P_*(u_n)\right] \to 0 \text{ as } n\to+\infty,$$

which ends the claim (i).

(ii) For every $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}$ and $\lambda \geq 0$, we have $\partial_t \widetilde{J}_\lambda(t(u,\lambda),u) = 0$ and $\partial_{tt} \widetilde{J}_\lambda(t(u,\lambda),u) < 0$, and $t(u,\lambda)$ is C^1 with respect to u. Let us introduce the C^1 functional $\mathcal{J}_\lambda$ defined on $\mathbb{S}$ by

$$\mathcal{J}_\lambda(u) = \widetilde{J}_\lambda(t(u,\lambda),u) \equiv J_\lambda(t(u,\lambda)u) = \mathcal{J}_\lambda(|u|).$$

Then

$$\alpha(\lambda) = \inf_{u \in \mathbb{S}} \mathcal{J}_\lambda(u).$$

Using the Ekeland variational principle on the complete manifold $(\mathbb{S}, \| \ \|)$ to the functional $\mathcal{J}_\lambda$, there exists a nonnegative minimizing sequence of (2) denoted by $(u_n) \subset \mathbb{S}$ such that:

$$|\mathcal{J}_\lambda'(u_n)(\varphi_n)| \leq \frac{1}{n}||\varphi_n||, \text{ for every } \varphi_n \in \mathrm{T}_{u_n}\mathbb{S},$$

where $\mathrm{T}_{u_n}\mathbb{S}$ is the tangent space to $\mathbb{S}$ at the point u_n. Moreover, for every $\varphi_n \in \mathrm{T}_{u_n}\mathbb{S}$, one has

$$\begin{aligned} \mathcal{J}_\lambda'(u_n)(\varphi_n) &= \partial_t \widetilde{J}_\lambda(t(u_n,\lambda),u_n)t'(u_n,\lambda)(\varphi_n) + \partial_u \widetilde{J}_\lambda(t(u_n,\lambda),u_n)(\varphi_n), \\ &= \partial_u \widetilde{J}_\lambda(t(u_n,\lambda),u_n)(\varphi_n), \end{aligned}$$

since $\partial_t \widetilde{J}_\lambda(t(u_n,\lambda),u_n) \equiv 0$, where $t'(u_n,\lambda)$ denotes the derivative of $t(.,\lambda)$ with respect to its first variable at the point (u_n,λ).
Furthermore, let

$$\begin{aligned} \pi \ : \ \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\} &\longrightarrow \mathbb{R} \times \mathbb{S} \\ u &\longmapsto \left(||u||, \frac{u}{||u||} \right) := (\pi_1(u), \pi_2(u)). \end{aligned}$$

Writing $\|u\| := \displaystyle\sum_{i=1}^N |\partial_{x_i} u|_{p_i}$ and applying Hölder's inequality for each $i \in \{1,2,\cdots N\}$, we get for every $(u,\varphi) \in \big(\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}\big) \times \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$:

$$\begin{cases} |\pi_1'(u)(\varphi)| & \leq \ ||\varphi||, \\ ||\pi_2'(u)(\varphi)|| & \leq \ 2\dfrac{||\varphi||}{||u||}. \end{cases}$$

From (i), there is a positive constant c_5 such that

$$t(u_n, \lambda) \geq c_5, \quad \forall\, n \in \mathbb{N}.$$

Then for every $\varphi \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$, there are $\varphi_n^1 := \pi_1'(t(u_n, \lambda)u_n)(\varphi) \in \mathbb{R}$ and $\varphi_n^2 := \pi_2'(t(u_n, \lambda)u_n)(\varphi) \in \mathrm{T}_{u_n}\mathbb{S}$ such that $|\varphi_n^1| \leq ||\varphi||$, $||\varphi_n^2|| \leq \dfrac{2}{c_5}||\varphi||$ and

$$\begin{aligned} J_\lambda'(t(u_n, \lambda)u_n)(\varphi) &= \partial_t \widetilde{J}_\lambda(t(u_n, \lambda), u_n)(\varphi_n^1) + \partial_u \widetilde{J}_\lambda(t(u_n, \lambda), u_n)(\varphi_n^2), \\ &= \partial_u \widetilde{J}_\lambda(t(u_n, \lambda), u_n)(\varphi_n^2), \\ &= \mathcal{J}_\lambda'(u_n)(\varphi_n^2). \end{aligned} \tag{6}$$

Therefore,

$$\begin{aligned} J_\lambda'(t(u_n, \lambda)u_n)(\varphi) &\leq \frac{1}{n}||\varphi_n^2|| \\ &\leq \frac{2}{nc_5}||\varphi||. \end{aligned}$$

We easily conclude that

$$\lim_{n\to\infty} J_\lambda'(t(u_n, \lambda)u_n) = 0 \ \text{ in } \ \mathcal{D}^{-1,\vec{p}'}(\mathbb{R}^N) = \left(\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)\right)',$$

which achieves the proof. □

Let us introduce the "critical level" c^* defined by

$$c^* := \inf_{u\in\mathbb{S}} J_0(t(u,0)u) = \inf_{N_0} \widetilde{J}_0(v). \tag{7}$$

Then, one has the

Proposition 2. *The critical level c^* satisfies:*

$$c^* = \inf_{N_0} \widetilde{J}_0(v) = \inf_{\widetilde{N}_0} \widetilde{J}_0(v).$$

Proof.

We know already that $c^* = \inf_{N_0} \widetilde{J}_0(v)$. For $v \in N_0$, $\sum_{i=1}^{N} P_i(v) = P_*(v)$ so that $J_0(v) = \widetilde{J}_0(v)$. Moreover, one has :

$$\inf_{\widetilde{N}_0} \widetilde{J}_0(v) \leq \inf_{N_0} \widetilde{J}_0(v) = c^* \leq \inf_{\widetilde{N}_0} \widetilde{J}_0(t(v,0)v) \leq \inf_{\widetilde{N}_0} \widetilde{J}_0(v), \ \ (\textit{since } t(v,0) \leq 1).$$

□

4 Existence results

At this stage, we will use a recent result about compactness for quasilinear Leray-Lions type operators involving critical exponents [6]. For the reader's convenience, we will recall some notations and the main result therein.
Let Ω be an arbitrary open set of $\mathbb{R}^N$. We set

$$\mathbb{L}^{\vec{p}}_{\text{loc}}(\Omega) = \prod_{i=1}^{N} L^{p_i}_{\text{loc}}(\Omega), \ \vec{p} = \big(p_1, \dots, p_N\big),$$

$$W^{1,\vec{p}}_{\text{loc}}(\Omega) = \Big\{ v \in \bigcup_{i=1}^{N} L^{p_i}_{\text{loc}}(\Omega) : \nabla v \in \mathbb{L}^{\vec{p}}_{\text{loc}}(\Omega) \Big\}.$$

For every $\varepsilon > 0$ and $\sigma \in \mathbb{R}$, consider the troncature function

$$S_\varepsilon(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| \le \varepsilon \\ \varepsilon \, \text{sign}(\sigma) & \text{otherwise.} \end{cases}$$

and set $\sigma^k := S_k(\sigma)$ for every integer k.
Let $\widehat{a} \ : \ \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ be a nonlinear map satisfying the standard Leray-Lions conditions :

(L1) $\widehat{a}(x, \cdot, \cdot)$ is continuous for almost every x and $\widehat{a}(\cdot, \sigma, \xi)$ is measurable for all $(\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

(L2) $\widehat{a}$ maps bounded sets of $W^{1,\vec{p}}_{\text{loc}}(\Omega)$ into bounded sets of $\prod_{i=1}^{N} L^{p_i'}_{\text{loc}}(\Omega)$.
For almost every $x \in \Omega$ and all $\xi \in W^{1,\vec{p}}_{\text{loc}}(\Omega)$, the map $u \longmapsto \widehat{a}(x, u, \xi)$ is continuous from $W^{1,\vec{p}}_{\text{loc}}(\Omega)$–weak to $\prod_{i=1}^{N} L^{p_i'}_{\text{loc}}(\Omega)$–strong.
For almost all $x \in \Omega$, for all (σ, ξ) in $\mathbb{R} \times \mathbb{R}^N$, $\widehat{a}(x, \sigma, \xi) \cdot \xi \ge 0$,

(L3) for almost every $x \in \Omega$, for all $(\sigma, \xi_i) \in \mathbb{R} \times \mathbb{R}^N, \ i = 1, 2,$

$$\big[\widehat{a}(x, \sigma, \xi_1) - \widehat{a}(x, \sigma, \xi_2)\big] \big[\xi_1 - \xi_2\big] > 0, \ for \ \xi_1 \ne \xi_2.$$

(L4) if for some $x \in \Omega$, there is a sequence $(\sigma_n, \xi_{1n}) \in \mathbb{R} \times \mathbb{R}^N$, $\xi_2 \in \mathbb{R}^N$ such that $\Big[\widehat{a}\big(x, \sigma_n, \xi_{1n}\big) - \widehat{a}\big(x, \sigma_n, \xi_2\big)\Big]\big[\xi_{1n} - \xi_2\big]$ and σ_n are bounded as then $|\xi_{1n}|$ remains in a bounded set of $\mathbb{R}$.

Theorem 1. **([6])**
Let (u_n) *be a bounded sequence of* $W^{1,\vec{p}}_{\mathrm{loc}}(\Omega)$. *Then*

(**i**) *There is a subsequence, still denoted by,* (u_n) *and a function* $u \in W^{1,\vec{p}}_{\mathrm{loc}}(\Omega)$ *such that* $u_n(x) \longrightarrow u(x)$ a.e. *in* Ω *as* $n \longrightarrow +\infty$.

(**ii**) *If furthermore, we assume (L1)-(L4) and that* $\forall\, \varphi \in C_c^\infty(\Omega)$, $\forall\, k \geq k_0 > 0$:

$$\limsup_{n \to +\infty} \int_\Omega \widehat{a}\big(x, u_n(x), \nabla u_n(x)\big) \cdot \nabla\big(\varphi S_\varepsilon(u_n - u^k)\big)\, dx \leq o_\varepsilon(1)$$

then there exists a subsequence still denoted by (u_n) *such that*

$$\nabla u_n(x) \longrightarrow \nabla u(x) \ \text{a.e. in } \Omega.$$

Corollary 1 (of Theorem 1). *Let* $q \in (p_+, p^*)$ *and* $\lambda \geq 0$. *Let* (u_n) *be a bounded Palais-Smale sequence for the functional* J_λ, *whose weak limit, up to a subsequence, is* u. *Then there exists a subsequence, still denoted by* (u_n) *such that*

$$\nabla u_n(x) \longrightarrow \nabla u(x) \ \text{a.e. in } \mathbb{R}^N.$$

Proof. It is a direct consequence of the above theorem. Indeed, since $(u_n)_{n \geq 0}$ remains in a bounded set of $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ then is bounded in $W^{1,\vec{p}}_{loc}(\mathbb{R}^N)$. Thus, passing if necessary to a subsequence, one has $u_n(x) \to u(x)$ as $n \to +\infty$ a.e. Setting

$$\widetilde{a}(\nabla u) = \left(\left|\frac{\partial u}{\partial x_1}\right|^{p_1 - 2} \frac{\partial u}{\partial x_1}, \ldots, \left|\frac{\partial u}{\partial x_N}\right|^{p_N - 2} \frac{\partial u}{\partial x_N} \right)$$

then, one has for every $\varphi \in C_c^\infty(\mathbb{R}^N)$:

$$J'_\lambda(u_n).\big(\varphi S_\varepsilon|u_n - u^k|\big) = \int_{\mathbb{R}^N} \widetilde{a}\big(\nabla u_n(x)\big) \cdot \nabla\big(\varphi S_\varepsilon(u_n - u^k)\big)dx$$
$$- \int_{\mathbb{R}^N} |u_n|^{p^*-2} u_n \varphi S_\varepsilon(u_n - u^k)dx - \lambda \int_{\mathbb{R}^N} a(x)|u_n|^{q-2} u_n \varphi S_\varepsilon(u_n - u^k).$$

Since (u_n) is bounded in $L^{p^*}(\mathbb{R}^N)$, there is a positive constant C such that $|u_n|_{a,q} \leq C$ and

$$\int_{\mathbb{R}^N} |u_n|^{p^*-1}|\varphi|\,|S_\varepsilon(u_n - u^k)| \leq \varepsilon|\varphi|_{L^{p^*}}\,|u_n|^{p^*-1}_{L^{p^*}} \leq C\,\varepsilon,$$

$$\int_{\mathbb{R}^N} a(x)|u_n|^{q-1}|\varphi|\,|S_\varepsilon| \leq C\,\varepsilon.$$

Since $\lim_n \|J'_\lambda(u_n)\|_* = 0$, we get

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} \widetilde{a}\big(\nabla u_n(x)\big) \cdot \nabla\big(\varphi S_\varepsilon(u_n - u^k)\big)dx \leq C\varepsilon.$$

□

Lemma 8. *We have the following assertions:*
(a) $\alpha(0) > 0$
(b) *For every* $\lambda > 0$, $\alpha(\lambda) \geq 0$.

Proof. (a) Let $(u_n) \subset \mathbb{S}$ be a minimizing sequence of (2) for $\lambda = 0$. Then, since $\sum_{i=1}^N P_i(u_n) = 1$, we get

$$\begin{aligned} J_0(t(u_n,0)u_n) &= \sum_{i=1}^N \left(\frac{1}{p_i} - \frac{1}{p^*}\right) t(u_n,0)^{p_i} P_i(u_n), \\ &\geq F(t(u_n,0)), \end{aligned}$$

where

$$F(t) = \begin{cases} \left(\dfrac{1}{p_+} - \dfrac{1}{p^*}\right) t^{p_+} & \text{if } t \in [0,1), \\ \left(\dfrac{1}{p_+} - \dfrac{1}{p^*}\right) t^{p_-} & \text{if } t \in [1,+\infty). \end{cases}$$

On the other hand, it is known from Lemma 7 that there is a constant $c_1 > 0$ such that $t(u_n, 0) \geq c_1 > 0$, for every integer n. Hence there exits is a constant $c_2 > 0$ such that $J_0(t(u_n, 0)u_n) \geq c_2 > 0$ for every integer n.
(b) Let $\lambda > 0$ and $u \in D^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}$. From Lemma 2, the positive real number $t(u, \lambda)$ is the unique value of t realizing the global maximum of the the real valued function φ_u: $t \longmapsto J_\lambda(tu)$ defined on $[0, +\infty)$. Since $\varphi_u(0) = 0$ then $\varphi_u(t(u, \lambda)) = J_\lambda(t(u, \lambda)u) \geq 0$. This implies that J_λ is nonnegative on the Nehari manifold $\mathcal{N}_{J_\lambda}$. □

Lemma 9. *Let (v_n) be a bounded sequence in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ satisfying:*

$$\sum_{i=1}^{N} P_i(v_n) = P_*(v_n) + o_n(1)$$

and

$$\lim_{n\to+\infty} \left(\sum_{i=1}^{N} P_i(v_n) \right) = \lim_{n\to+\infty} P_*(v_n) = b \neq 0.$$

Then

$$\lim_{n\to+\infty} t(v_n, 0) = 1.$$

Proof. Actually, for every $u \in D^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}$, the quantity $t(u, 0)$ depends on u only by the mean of $P_i(u)$, $1 \leq i \leq N$, and $P_*(u)$. Then let us write

$$\forall\, u \in D^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}, \;\; t(u, 0) := \tau(P_1(u), P_2(u), \cdots, P_N(u), P_*(u)).$$

We extend the domain of definition of τ to $[0, +\infty)^N \times]0, +\infty)$ as the following: For every $(\beta_1, \beta_2, \cdots, \beta_{N+1}) \in [0, +\infty)^N \times]0, +\infty)$, we define $\tau(\beta_1, \beta_2, \cdots, \beta_{N+1})$ by substituting $(P_1(u), P_2(u), \cdots, P_N(u), P_*(u))$ by $(\beta_1, \beta_2, \cdots, \beta_{N+1})$ in the definition of $\tau(P_1(u), P_2(u), \cdots, P_N(u), P_*(u))$. Moreover, since $t(u, 0) = 1$ for every $u \in \mathcal{N}_{J_0}$, it follows that for every $u \in D^{1,\vec{p}}(\mathbb{R}^N) \setminus \{0\}$, if $\sum_{i=1}^{N} P_i(u) = P_*(u)$ then $\tau(P_1(u), P_2(u), \cdots, P_N(u), P_*(u)) = 1$. Therefore, for every

$(\beta_1, \beta_2, \cdots, \beta_{N+1}) \in [0, +\infty)^N \times]0, +\infty)$, it holds

$$\sum_{i=1}^{N} \beta_i = \beta_{N+1} \Longrightarrow \tau(\beta_1, \beta_2, \cdots, \beta_{N+1}) = 1.$$

On the other hand, for every $i \in \{1, \cdots, N\}$, the sequences $(P_i(v_n))_n$ is bounded in $\mathbb{R}$. If $\beta_i \geq 0$ is an arbitrary adherence value of $(P_i(v_n))_n$, for $1 \leq i \leq N$, then

$$\sum_{i=1}^{N} \beta_i = \lim_{n \to +\infty} P_*(v_n) = b \neq 0,$$

and consequently $\tau(\beta_1, \cdots, \beta_N, b) = 1$. Since $t(u, 0)$ depends continuously on $(P_1(u), \cdots, P_N(u), P_*(u))$ in $\mathbb{R}^{N+1}$, it follows that $\lim_{n\to+\infty} t(v_n, 0) = \tau(\beta_1, \cdots, \beta_N, b) = 1$, which ends the proof. $\square$

Lemma 10. *Let $q \in (p_+, p^*)$ and $\lambda \geq 0$. Let (u_n) be a (PS)$_c$ for the functional J_λ such that $c < c^*$, then (u_n) is relatively compact.*

Proof. Let (u_n) be a (PS)$_c$ for the functional J_λ such that $c < c^*$, with $\lambda \geq 0$. We can show easily that (u_n) is bounded. According to Corollary 1 of Theorem 1 we can assume then that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N), \\ u_n &\to u \text{ a.e. in } \mathbb{R}^N, \\ \nabla u_n(x) &\to \nabla u(x) \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

We set $v_n := u_n - u$, then using the Brézis-Lieb Lemma, it follows that

$$\begin{aligned} \sum_{i=1}^{N} P_i(v_n) &= P_*(v_n) + \mathrm{o}_n(1), \\ J_0(v_n) = &= c - J_\lambda(u) + \mathrm{o}_n(1). \end{aligned}$$

Let b be the common limit to $\sum_{i=1}^{N} P_i(v_n)$ and $P_*(v_n)$ as n goes to infinity. Suppose that $b \neq 0$. Applying Lemma 9, it follows that $\lim_{n\to+\infty} t(v_n, 0) = 1$

and consequently

$$\lim_{n\to+\infty} J_0(t(v_n,0)v_n) = \lim_{n\to+\infty} J_0(v_n) = c - J_\lambda(u).$$

On the other hand,

$$J_0(t(v_n,0)v_n) \geq \inf_{w\in\mathcal{N}_{J_0}} J_0(w).$$

Therefore, $c - J_\lambda(u) \geq c^*$, which means that $c \geq c^*$ since $u \in \mathcal{N}_{J_\lambda}$ and it is known that J_λ is nonnegative on $\mathcal{N}_{J_\lambda}$. This achieves the proof. □

Let $M^* = \Big\{v \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)\backslash\{0\} : Q(v) > 0 \text{ and } \widetilde{J}_0(v) = c^*\Big\}$. Since $c^* = \alpha(0) > 0$, thus M^* is not empty. We define :

$$\lambda^* = \inf\Big\{\lambda_{min}(u) : u \in M^*\Big\}$$

we have the following fundamental proposition :

Proposition 3. *If* $\lambda > \lambda^* \geq 0$ *then* $\alpha(\lambda) < c^*$.

Proof. Let $\lambda > \lambda^*$, then there exists $u \in M^*$ such that $\lambda > \lambda_{min}(u) > \lambda^*$, thus $t(u,\lambda) \leq 1$. From Lemma 6, one has

$$J_\lambda\big(t(u,\lambda)u\big) = \widetilde{J}_0\big(t(u,\lambda)u\big) - \lambda t^q(u,\lambda)\left(\frac{1}{q} - \frac{1}{p^*}\right)Q(u). \tag{8}$$

Since $\widetilde{J}_0\big(t(u,\lambda)u\big) \leq \widetilde{J}_0(u) = c^*$, the above relation (8) becomes

$$\alpha(\lambda) \leq J_\lambda\big(t(u,\lambda)u\big) \leq c^* - \lambda t(u,\lambda)^q\left(\frac{1}{q} - \frac{1}{p^*}\right)Q(u) < c^*.$$

□

At this stage, we state and show our main result:

Theorem 2. *For every* $\lambda > \lambda^*$*, there exists at least one nonnegative non-trivial solution to (1).*

Proof. From Lemma 7 there is a sequence $U_n = t(u_n, \lambda)u_n \geq 0$ bounded in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ and which is a Palais sequence for J_λ at the level $\alpha(\lambda)$, i.e.

$$J_\lambda(U_n) \to \alpha(\lambda), \qquad \liminf_{n\to+\infty} \|U_n\| > 0, \text{ as } n \to +\infty,$$

and

$$J'_\lambda(U_n) \to 0 \ in \ \left(\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)\right)' \text{ as } n \to +\infty.$$

Passing, if necessary to a subsequence, we may assume that $U_n \rightharpoonup U$ in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$-weak, a.e. in $\mathbb{R}^N$. If $\lambda > \lambda^*$ then $\alpha(\lambda) < c^*$, thus applying Lemma 10, one deduces that $U_n \to U$ in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$-strong and $L^{p^*}(\mathbb{R}^N)$-strong. Thus

$$\begin{cases} J_\lambda(U) = \alpha(\lambda) \geq 0 : U \in \mathcal{D}^{1,\vec{p}}_+(\mathbb{R}^N)\backslash\{0\}, \\ J'_\lambda(U) = 0, \\ \|U\| = \lim\limits_n \|U_n\| > 0. \end{cases}$$

□

Remark. If $u \in M^* \cap \widetilde{N}_0$ then $\lambda_{min}(u) = 0$ thus $\lambda^* = 0$.

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