The Nehari manifold for systems of nonlinear elliptic equations

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Abstract

This paper deals with existence and multiplicity results of nonlocal positive solutions to the following system

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p_1-2}u + (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v &= \mu |v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v, \end{cases}$$

together with Dirichlet or mixed boundary conditions, under some hypotheses on the parameters p, p_1, α, β and q. More precisely, the system considered corresponds to a perturbed eigenvalue equation combined with a second equation having concave and convex nonlinearities. The study is based on the extraction of Palais-Smale sequences in the Nehari manifold. The behaviour of the energy corresponding to these positive solutions, with respect to the real parameters λ and μ, is established.

1 Introduction

In this work, we consider the system of quasilinear elliptic equations

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p_1-2}u + (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v &= \mu |v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v, \end{cases} \tag{1.1}$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}|_{\Sigma_1} = 0, \\ v|_{\Gamma_2} = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0, \end{cases} \tag{1.2}$$

where, Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega = \overline{\Gamma}_i \cap \overline{\Sigma}_i$, where Γ_i are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma_i \cap \Sigma_i = \emptyset$, $i \in \{1, 2\}$. Δ_p is the p-Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative.

It is clear that when $\Gamma_1 = \Gamma_2 = \partial\Omega$, one deals with homogeneous Dirichlet boundary conditions.

Our aim here is to establish nonlocal existence and multiplicity results, with respect to the real parameters λ and μ, for Problem (1.1). Along this work, the following assumptions will hold

$$1 < p_1 < p < N, \quad q > 1, \quad \alpha > 1, \quad \beta > 1, \tag{1.3}$$

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1, \tag{1.4}$$

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1 \text{ and } \frac{\beta+1}{q} < 1, \tag{1.5}$$

where

$$p^* = \frac{Np}{N-p}, \quad q^* = \frac{Nq}{N-q}$$

are the critical exponents for the p-Laplacian and q-Laplacian respectively. These assumptions mean that we are concerned with a subcritical and super-homogeneous system where the first equation is *concave-convex* and the second equation is only a perturbation of an eigenvalue equation. Also, the following assumptions concerning the real parameters λ and μ will hold

$$\lambda > 0, \quad \mu < \mu_1,$$

where μ_1 is the first eigenvalue of $-\Delta_q$ in Ω.

Problem (1.1), together with (1.2), is posed in the framework of the Sobolev space $W = W^{1,p}_{\Gamma_1}(\Omega) \times W^{1,q}_{\Gamma_2}(\Omega)$, where

$$W^{1,p}_{\Gamma_1}(\Omega) = \{u \in W^{1,p}(\Omega) \ : \ u|_{\Gamma_1} = 0\}, \ \ W^{1,q}_{\Gamma_2}(\Omega) = \{u \in W^{1,q}(\Omega) \ : \ u|_{\Gamma_2} = 0\},$$

are respectively the closure of $C^1_0(\Omega \cap \Gamma_1, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$ and $C^1_0(\Omega \cap \Gamma_2, \mathbb{R})$ with respect to the norm of $W^{1,q}(\Omega)$. We can refer the reader to [9] for a complete description of this space in the case $p = 2$. Notice that $meas(\Gamma_i) > 0$, $i = 1, 2$, imply that the Poincaré inequality is still available in $W^{1,p}_{\Gamma_1}(\Omega)$ and $W^{1,q}_{\Gamma_2}(\Omega)$, so W can be endowed with the norm

$$||(u, v)|| = ||\nabla u||_p + ||\nabla v||_q$$

and $(W, ||\,.\,||)$ is a reflexive and separable Banach space.

Semilinear and quasilinear scalar elliptic equations with concave and convex nonlinearities are widely studied, we can refer the reader to [1, 4, 10, 18] and to the survey article [5]. For the nonlinear elliptic systems, we refer to [2, 3, 6, 8, 11, 14, 20, 21] and to the survey article [13]. In [15], the authors studied the existence of positive solutions to a perturbed eigenvalue problem involving the p-Laplacian operator. In [6], the authors have generalized the results of [15] to a perturbed eigenvalue system involving p and q-Laplacian operators. Recently, in [10] the first author has considered a semilinear elliptic equation with concave and convex nolinearities, and showed *nonlocal* existence and multiplicity results with respect to the parameter via the extraction of Palais-Smale sequences in the Nehari manifold.

In this paper, we extend this method to the system (1.1) where one equation contains concave and convex nonlinearities and the other one is simply a perturbation of an eigenvalue equation. We show that Problem (1.1) has at least two positive solutions when the pair of parameters (λ, μ) belongs to a subset of $\mathbb{R}^2$ which will be specified below.

For solutions of (1.1) we understand critical points of the Euler-Lagrange functional $I \in C^1(W, \mathbb{R})$ given by

$$I(u,v) = \frac{1}{p}P(u) - \frac{\lambda}{p_1}P_1(u) + \frac{1}{q}(Q(v) - \mu Q_1(v)) - R(u,v),$$

where $P(u) = ||\nabla u||_p^p$, $P_1(u) = ||u||_{p_1}^{p_1}$, $Q(v) = ||\nabla v||_q^q$, $Q_1(v) = ||v||_q^q$ and $R(u,v) = \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}dx$.

Consider the "Nehari" manifold [16] associated to Problem (1.1) given by

$$\mathcal{N} = \{(u,v) \in (W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\}) \;/\; D_1 I(u,v)(u) = D_2 I(u,v)(v) = 0\},$$

where $D_1 I$ and $D_2 I$ are the derivatives of I with respect to the first variable and the second variable respectively.

An interesting and useful characterization of $\mathcal{N}$, [15, 18, 22, 10, 7] is the following

$$\mathcal{N} = \{(su, tv) \;/\; (s,u,t,v) \in \mathcal{Z}^* \text{ and } \partial_s I(su,tv) = \partial_t I(su,tv) = 0\},$$

where $\mathcal{Z}^* = (\mathbb{R}\setminus\{0\}) \times (W^{1,p}_{\Gamma_1}(\Omega)\setminus\{0\}) \times (\mathbb{R}\setminus\{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega)\setminus\{0\})$ and I is considered as a functional of four variables (s,u,t,v) in $\mathcal{Z} := \mathbb{R} \times W^{1,p}_{\Gamma_1}(\Omega) \times \mathbb{R} \times W^{1,q}_{\Gamma_2}(\Omega)$. For this reason, we introduce the modified Euler-Lagrange functional $\widetilde{I}$ defined on $\mathcal{Z}$ by

$$\widetilde{I}(s,u,t,v) := I(su,tv).$$

2 Preliminary results

In this work, we are interested by nontrivial positive solutions $u \neq 0$ and $v \neq 0$ to Problem (1.1). Since the functional $\widetilde{I}$ is even in s and t, we limit our study for $s > 0$, $t > 0$ and for $(u, v) \in (W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\})$.

Lemma 2.1 *For every $(u, v) \in (W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\})$ there exists a unique $\lambda(u, v) > 0$ such that the real-valued function $(s, t) \in (0, +\infty)^2 \mapsto \widetilde{I}(s, u, t, v)$ has exactly two critical points (resp. one critical point) for $0 < \lambda < \lambda(u, v)$ (resp. $\lambda = \lambda(u, v)$). This functional has no critical point for $\lambda > \lambda(u, v)$.*

Proof. Let (u, v) be an arbitrary element in $(W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\})$. Then

$$\widetilde{I}(s, u, t, v) = \frac{s^p}{p} P(u) - \frac{\lambda}{p_1} s^{p_1} P_1(u) + \frac{t^q}{q} (Q(v) - \mu Q_1(v)) - s^{\alpha+1} t^{\beta+1} R(u, v).$$

A direct computation gives $\partial_t \widetilde{I}(s, u, t, v) = 0$ if and only if

$$t = t(s) = \left[(\beta + 1) \frac{R(u, v)}{Q(v) - \mu Q_1(v)} \right]^{\frac{1}{q-(\beta+1)}} s^{\frac{\alpha+1}{q-(\beta+1)}}, \tag{2.6}$$

and

$$\widetilde{I}(s, u, t(s), v) = \frac{s^p}{p} P(u) - \frac{\lambda}{p_1} s^{p_1} P_1(u) - \frac{s^r}{r} A(u, v),$$

where

$$A(u, v) = (\alpha + 1)(\beta + 1)^{\frac{\alpha+1}{q-(\beta+1)}} \frac{R(u, v)^{\frac{q}{q-(\beta+1)}}}{(Q(v) - \mu Q_1(v))^{\frac{\beta+1}{q-(\beta+1)}}}$$

and $r = \frac{(\alpha+1)q}{q-(\beta+1)}$. It is easy to verify that $r > p$. Now consider the function $s \in (0, +\infty) \mapsto \widetilde{I}(s, u, t(s), v)$ and let us write

$$\partial_s \widetilde{I}(s, u, t(s), v) := s^{p_1 - 1} F_{\lambda,\mu}(s, u, v).$$

where $F_{\lambda,\mu}(s, u, v) := P(u) s^{p-p_1} - \lambda P_1(u) - A(u, v) s^{r-p_1}$. The function $s \in (0, +\infty) \mapsto F_{\lambda,\mu}(s, u, v)$ is increasing on $(0, \overline{s}_\mu(u, v))$, decreasing on $(\overline{s}_\mu(u, v), +\infty)$ and attains its unique maximum for $s = \overline{s}_\mu(u, v)$, where

$$\overline{s}_\mu(u, v) = \left[\frac{p - p_1}{r - p_1} \frac{P(u)}{A(u, v)} \right]^{\frac{1}{r-p}}. \tag{2.7}$$

So, the function $s \in (0, +\infty) \mapsto F_{\lambda,\mu}(s, u, v)$ has two positive zeros (resp. one positive zero) if $F_{\lambda,\mu}(\overline{s}_\mu(u, v), u, v) > 0$ (resp. $F_{\lambda,\mu}(\overline{s}_\mu(u, v), u, v) = 0$) and has no zero if $F_{\lambda,\mu}(\overline{s}_\mu(u, v), u, v) < 0$. On the other hand, a direct computation leads to

$$F_{\lambda,\mu}(\overline{s}_\mu(u, v), u, v) = \frac{r - p}{r - p_1} \left[\frac{p - p_1}{r - p_1} \frac{P(u)}{A(u, v)} \right]^{\frac{p-p_1}{r-p_1}} P(u) - \lambda P_1(u).$$

Then, $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v) > 0$ (resp. $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v) < 0$) if $\lambda < \lambda(u,v)$ (resp. $\lambda > \lambda(u,v)$) and $F_{\lambda(u,v),\mu}(\overline{s}_\mu(u,v),u,v) = 0$, where

$$\lambda(u,v) = \widehat{c}\frac{P(u)^{\frac{r-p_1}{r-p}}}{P_1(u)A(u,v)^{\frac{p-p_1}{r-p}}} \quad \text{and} \quad \widehat{c} = \frac{r-p}{r-p_1}\left[\frac{p-p_1}{r-p_1}\right]^{\frac{p-p_1}{r-p}}. \tag{2.8}$$

Therefore, if $\lambda \in (0,\lambda(u,v))$, the function $s \in (0,+\infty) \mapsto \partial_s \widetilde{I}(s,u,t(s),v)$ has two positive zeros denoted by $s_1(u,v,\lambda,\mu)$ and $s_2(u,v,\lambda,\mu)$ verifying $0 < s_1(u,v,\lambda,\mu) < \overline{s}_\mu(u,v) < s_2(u,v,\lambda,\mu)$. Since $F_{\lambda,\mu}(s_1(u,v,\lambda,\mu),u,v) = F_{\lambda,\mu}(s_2(u,v,\lambda,\mu),u,v) = 0$, $\partial_s F_{\lambda,\mu}(s,u,v) > 0$ for $0 < s < \overline{s}_\mu(u,v)$ and $\partial_s F_{\lambda,\mu}(s,u,v) < 0$ for $s > \overline{s}_\mu(u,v)$ it follows that

$$\partial_{ss}\widetilde{I}(s_1(u,v,\lambda,\mu),u,t(s_1(u,v,\lambda,\mu)),v) \;>\; 0, \tag{2.9}$$

$$\partial_{ss}\widetilde{I}(s_2(u,v,\lambda,\mu),u,t(s_2(u,v,\lambda,\mu)),v) \;<\; 0. \tag{2.10}$$

This implies that the real-valued function $s \in (0,+\infty) \mapsto \widetilde{I}(s,u,t(s),v)$ achieves its unique local minimum at $s = s_1(u,v,\lambda,\mu)$ and its unique local maximum at $s = s_2(u,v,\lambda,\mu)$, which ends the proof. $\square$

Hereafter, we will denote $t_i(u,v,\lambda,\mu) := t(s_i(u,v,\lambda,\mu))$, $i = 1, 2$. At this stage, we introduce the characteristic value

$$\widehat{\lambda}(\mu) := \inf\{\, \lambda(u,v),\ (u,v) \in (W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\})\,\}.$$

We claim that $\widehat{\lambda}(\mu)$ is great than a positive constant which depends only on μ, p, p_1, q, α, β and Ω. Indeed, using the Hölder inequality, we get

$$R(u,v) \le |\Omega|^{\delta} ||u||_{p^*}^{\alpha+1} ||v||_{q^*}^{\beta+1},$$

where $\delta > 1$ is such that $\frac{1}{p^*} + \frac{1}{q^*} + \frac{1}{\delta} = 1$. Using the continuous embedding $W^{1,q}_{\Gamma_2}(\Omega) \subset L^{q^*}(\Omega)$ we get

$$A(u,v) \le c_1 \frac{P_*(u)^{\frac{r}{p^*}}}{(\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)}}},$$

where $P_*(u) = ||u||_{p^*}^{p^*}$ and $c_1 = c_1(p,p_1,q,\alpha,\beta,\Omega)$. Using again the continuous embeddings $W^{1,p}_{\Gamma_1}(\Omega) \subset L^{p_1}(\Omega)$ and $W^{1,p}_{\Gamma_1}(\Omega) \subset L^{p^*}(\Omega)$ we obtain

$$\lambda(u,v) \ge c_2(\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)}\frac{p-p_1}{r-p}},$$

where $c_2 = c_2(p, p_1, q, \alpha, \beta, \Omega)$ and then

$$\widehat{\lambda}(\mu) \geq c_2(\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)}\frac{p-p_1}{r-p}},$$

which achieves the claim. Now let us introduce

$$\mathcal{D} := \{(\lambda, \mu) \in (0, +\infty) \times (-\infty, \mu_1) \ : \ \lambda < \widehat{\lambda}(\mu)\}.$$

For every $(\lambda, \mu) \in \mathcal{D}$, the functionals $(u, v) \in (W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\}) \mapsto \widetilde{I}(s_i(u, v, \lambda, \mu), u, t_i(u, v, \lambda, \mu), v)$ $i = 1, 2$, are well defined and one can show easily that they are bounded below. Hence, for every $(\lambda, \mu) \in \mathcal{D}$, we define

$$\alpha_1(\lambda, \mu) \ := \ \inf\{\widetilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v), \ (u, v) \in \widetilde{W}\} \tag{2.11}$$
$$\alpha_2(\lambda, \mu) \ := \ \inf\{\widetilde{I}(s_2(u, v, \lambda, \mu), u, t_2(u, v, \lambda, \mu), v), \ (u, v) \in \widetilde{W}\} \tag{2.12}$$

where

$$\widetilde{W} := (W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\}).$$

Our aim in the sequel is to show that $\alpha_1(\lambda, \mu)$ and $\alpha_2(\lambda, \mu)$ are in fact critical values of the Euler-Lagrange functional I for every $(\lambda, \mu) \in \mathcal{D}$. We start with the following

Lemma 2.2 *Let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11) (resp. of (2.12)) and let $(U^1_n, V^1_n) := (s_1(u_n, v_n, \lambda, \mu)u_n, t_1(u_n, v_n, \lambda, \mu)v_n)$ (resp. $(U^2_n, V^2_n) := (s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$). Then it holds:*

(i) $\limsup_{n\to+\infty} ||(U^1_n, V^1_n)|| < \infty$ $\quad$ (*resp.* $\limsup_{n\to+\infty} ||(U^2_n, V^2_n)|| < \infty$).

(ii) $\liminf_{n\to+\infty} ||(U^1_n, V^1_n)|| > 0$ $\quad$ (*resp.* $\liminf_{n\to+\infty} ||(U^2_n, V^2_n)|| > 0$).

Proof. We show the assertion (i), let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11). Since $\partial_s \widetilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$ and $\partial_t \widetilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$, it follows that

$$P(U^1_n) - \lambda P_1(U^1_n) - (\alpha + 1)R(U^1_n, V^1_n) \ = \ 0, \tag{2.13}$$
$$Q(V^1_n) - \mu Q_1(V^1_n) - (\beta + 1)R(U^1_n, V^1_n) \ = \ 0. \tag{2.14}$$

Suppose that there is a subsequence, still denoted by (U^1_n, V^1_n), such that $\lim_{n\to+\infty} ||(U^1_n, V^1_n)|| = \infty$. We will distinguish three cases:

Case a) $\lim_{n\to+\infty} ||\nabla U^1_n||_p = \infty$ and $||\nabla V^1_n||_q$ is bounded. By (2.14) we get that $R(U^1_n, V^1_n)$ is bounded. On the other hand, using the continuous embedding

$W^{1,p}_{\Gamma_1}(\Omega) \subset L^{p_1}(\Omega)$, we have $P_1(U^1_n) = o_n\left(P(U^1_n)\right)$, as n goes to $+\infty$. By (2.13) we get $R(U^1_n, V^1_n) = \frac{1}{\alpha+1}(1 + o_n(1))P(U^1_n)$ as n goes to $+\infty$ and hence $\lim_{n\to+\infty} R(U^1_n, V^1_n) = +\infty$, which cannot hold true.

Case b) $\lim_{n\to+\infty} ||\nabla V^1_n||_q = \infty$ and $||\nabla U^1_n||_p$ is bounded. By (2.13) we get $R(U^1_n, V^1_n)$ bounded. If $0 < \mu < \mu_1$, using the Sobolev and Young inequalities, for every $\varepsilon \in (0, 1)$, there is a positive constant C_ε such that

$$||V^1_n||^q_q \leq \frac{\varepsilon}{\mu} ||\nabla V^1_n||^q_q + C_\varepsilon,$$

which gives $(\beta+1)R(U^1_n, V^1_n) + \mu C_\varepsilon \geq (1-\varepsilon)Q(V^1_n)$. Then $\lim_{n\to+\infty} R(U^1_n, V^1_n) = +\infty$, which is impossible. If $\mu < 0$, then $Q(V^1_n) - \mu Q_1(V^1_n) = (\beta + 1)R(U^1_n, V^1_n) \geq Q(V^1_n)$ so $\lim_{n\to+\infty} R(U^1_n, V^1_n) = +\infty$, which is also impossible.

Case c) $\lim_{n\to+\infty} ||\nabla U^1_n||_p = \lim_{n\to+\infty} ||\nabla V^1_n||_q = \infty$. As in the first case, we have

$$R(U^1_n, V^1_n) = \frac{1}{\alpha + 1}(1 + o_n(1))P(U^1_n), \text{ as } n \text{ goes to } +\infty.$$

Then $I(U^1_n, V^1_n) = \frac{1}{\alpha+1}\left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1 + o_n(1)\right) P(U^1_n)$ as n goes to $+\infty$. Hence, using the hypothese (1.5), $\lim_{n\to+\infty} I(U^1_n, V^1_n) = +\infty$, which is impossible. Consequently, $\limsup_{n\to+\infty} ||(U^1_n, V^1_n)|| < \infty$. We show in the same way that $\limsup_{n\to+\infty} ||(U^2_n, V^2_n)|| < \infty$.

Now, we show the assertion (ii), let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11). Suppose that there is a subsequence, still denoted by (U^1_n, V^1_n), such that $\lim_{n\to+\infty} ||(U^1_n, V^1_n)|| = 0$. By (2.13) we get $\lim_{n\to+\infty} I(U^1_n, V^1_n) = 0$ and this can not hold true because $I(U^1_n, V^1_n) < 0$ for every n.

Similarly, let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.12). Suppose that there is a subsequence, still denoted by (U^2_n, V^2_n), such that
$\lim_{n\to+\infty} ||(U^2_n, V^2_n)|| = 0$. If $p > \alpha + 1$, by (2.10) , we have

$$\partial_{ss} I(U^2_n, V^2_n) = (p - 1)P(U^2_n) - \lambda(p_1 - 1)P_1(U^2_n) - \alpha(\alpha + 1)R(U^2_n, V^2_n) < 0$$

Then $(p - 1)P(U^2_n) - \lambda(p - 1)P_1(U^2_n) - \alpha p R(U^2_n, V^2_n) < 0$, which implies that $(p - (\alpha + 1))R(U^2_n, V^2_n) < 0$ and this is impossible. Finally, if $p \leq \alpha + 1$, then $(p - p_1)P(U^2_n) < (\alpha + 1)^2 R(U^2_n, V^2_n)$. Since $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$, then there exist $\tilde{p}$ and $\tilde{q}$ satisfying $p < \tilde{p} < p^*$, $q < \tilde{q} < q^*$ and

$$\frac{\alpha + 1}{\tilde{p}} + \frac{\beta + 1}{\tilde{q}} = 1. \tag{2.15}$$

Therefore,

$$\begin{aligned} R(U_n^2, V_n^2) &\leq c(\Omega, p, q)||U_n^2||_{\tilde{p}}^{\alpha+1}||V_n^2||_{\tilde{q}}^{\beta+1} \\ &\leq c'(\Omega, p, q)||\nabla U_n^2||_p^{\alpha+1}||\nabla V_n^2||_q^{\beta+1} \end{aligned}$$

and consequently, $(p-p_1) \leq c'(\Omega, p, q)(\alpha+1)^2||\nabla U_n^2||_p^{\alpha+1-p}||\nabla V_n^2||_q^{\beta+1}$ which converges to 0 as n goes to $+\infty$. This contradicts the fact $p > p_1$, which ends the proof. □

3 Palais-Smale sequences in the Nehari Manifold

It is interesting to notice that for every $\gamma > 0$, $\delta > 0$, it holds

$$\begin{aligned} \widetilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \widetilde{I}(s, u, t, v), \\ \partial_t \widetilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\delta}\partial_t \widetilde{I}(s, u, t, v), \\ \partial_s \widetilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\gamma}\partial_s \widetilde{I}(s, u, t, v), \\ \partial_{ss} \widetilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\gamma^2}\partial_{ss} \widetilde{I}(s, u, t, v). \end{aligned}$$

This implies that

$$s_1(u, v, \lambda, \mu) = \frac{1}{\gamma}s_1\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall\, \delta > 0, \tag{3.16}$$

$$s_2(u, v, \lambda, \mu) = \frac{1}{\gamma}s_2\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall\, \delta > 0, \tag{3.17}$$

$$t_1(u, v, \lambda, \mu) = \frac{1}{\delta}t_1\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall\, \gamma > 0, \tag{3.18}$$

$$t_2(u, v, \lambda, \mu) = \frac{1}{\delta}t_2\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall\, \gamma > 0. \tag{3.19}$$

It follows that

$$\alpha_1(\lambda, \mu) = \inf_{(u,v)\in\mathbb{S}_p\times\mathbb{S}_q}\{\widetilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v)\}, \tag{3.20}$$

$$\alpha_2(\lambda, \mu) = \inf_{(u,v)\in\mathbb{S}_p\times\mathbb{S}_q}\{\widetilde{I}(s_2(u, v, \lambda, \mu), u, t_2(u, v, \lambda, \mu), v)\}, \tag{3.21}$$

where $\mathbb{S}_p$ and $\mathbb{S}_q$ are the unit spheres of $W^{1,p}_{\Gamma_1}(\Omega)$ and $W^{1,q}_{\Gamma_2}(\Omega)$ respectively. Make precise that $\mathbb{S}_p \times \mathbb{S}_q$ is a 2-codimensional and complete submanifold of W, we will denote it in the sequel by $\mathbb{S}$.

Lemma 3.1 *Let $(\lambda,\mu)\in\mathcal{D}$ and let $(u_n,v_n)\in\mathbb{S}$ be a minimizing sequence of (3.20) (resp. of (3.21)). Then $(s_1(u_n,v_n,\lambda,\mu)u_n, t_1(u_n,v_n,\lambda,\mu)v_n)$,*
(resp. $(s_2(u_n,v_n,\lambda,\mu)u_n, t_2(u_n,v_n,\lambda,\mu)v_n)$) is a Palais-Smale sequence for the functional I.

Proof. Let $(\lambda,\mu)\in\mathcal{D}$ and consider a minimizing sequence $(u_n,v_n)\in\mathbb{S}$ of (3.20). Let us set

$$\begin{aligned} U_n &= s_1(u_n,v_n,\lambda,\mu)u_n,\\ V_n &= t_1(u_n,v_n,\lambda,\mu)v_n. \end{aligned}$$

The sequence (U_n,V_n) is clearly bounded in W. On the other hand, the gradient (resp. the Hessian determinant) of $\widetilde{I}$ with respect to s and t at $(s,t) = (s_1(u_n,v_n,\lambda,\mu), t_1(u_n,v_n,\lambda,\mu))$ is equal to zero (resp. is strictly negative). So, the implicit function theorem implies that that $s_1(u_n,v_n,\lambda,\mu)$ and $t_1(u_n,v_n,\lambda,\mu)$ are C^1 with respect to (u,v), since $\widetilde{I}$ is.

We introduce now the functional $\mathcal{I}$ defined on $\mathbb{S}$ by

$$\mathcal{I}(u,v)=\widetilde{I}(s_1(u,v,\lambda,\mu),u,t_1(u,v,\lambda,\mu),v),$$

then

$$\alpha_1(\lambda,\mu)=\inf_{(u,v)\in\mathbb{S}}\mathcal{I}(u,v)=\lim_{n\to+\infty}\mathcal{I}(u_n,v_n).$$

Applying the Ekeland variational principle [12, 17, 19, 22] on the complete manifold $(\mathbb{S}, ||\,.\,||)$ to the functional $\mathcal{I}$ we get

$$\mathcal{I}'(u_n,v_n)(\varphi_n,\psi_n)\le\frac{1}{n}||(\varphi_n,\psi_n)||,\;\;\forall(\varphi_n,\psi_n)\in T_{(u_n,v_n)}\mathbb{S},$$

where $T_{(u_n,v_n)}\mathbb{S}$ denotes the tangent space to $\mathbb{S}$ at the point (u_n,v_n). Recall that $T_{(u_n,v_n)}\mathbb{S}=T_{u_n}\mathbb{S}_p\times T_{v_n}\mathbb{S}_q$, where $T_{u_n}\mathbb{S}_p$ (resp. $T_{v_n}\mathbb{S}_q$) is the tangent space to $\mathbb{S}_p$ (resp. $\mathbb{S}_q$) at the point u_n (resp. v_n).

Set

$$A_n:=(u_n,v_n,\lambda,\mu),\;\text{and}\;B_n:=(s_1(u_n,v_n,\lambda,\mu),u_n,t_1(u_n,v_n,\lambda,\mu),v_n).$$

For every $(\varphi_n,\psi_n)\in T_{u_n}\mathbb{S}_p\times T_{v_n}\mathbb{S}_q$, one has

$$\mathcal{I}'(u_n,v_n)(\varphi_n,\psi_n)=D_1\widetilde{I}(B_n)(\varphi_n)+D_2\widetilde{I}(B_n)(\psi_n)$$

where

$$\begin{aligned} D_1\widetilde{I}(B_n)(\varphi_n) &= \partial_u s_1(A_n)(\varphi_n)\partial_s\widetilde{I}(B_n) + \partial_u\widetilde{I}(B_n)(\varphi_n) + \partial_u t_1(A_n)(\varphi_n)\partial_t\widetilde{I}(B_n) \\ &= \partial_u\widetilde{I}(B_n)(\varphi_n). \end{aligned}$$

Similarly, one has

$$D_2\widetilde{I}(B_n)(\psi_n) = \partial_v\widetilde{I}(B_n)(\psi_n).$$

Furthermore, consider the "fiber" maps

$$\begin{aligned} \pi \ : \ W^{1,p}_{\Gamma_1}(\Omega)\setminus\{0\} &\longrightarrow \mathbb{R}\times\mathbb{S}_p \\ u &\longmapsto \left(||\nabla u||_p, \tfrac{u}{||\nabla u||_p}\right) := (\pi_1(u), \pi_2(u)), \\ \widetilde{\pi} \ : \ W^{1,q}_{\Gamma_2}(\Omega)\setminus\{0\} &\longrightarrow \mathbb{R}\times\mathbb{S}_q \\ v &\longmapsto \left(||\nabla v||_q, \tfrac{v}{||\nabla v||_q}\right) := (\widetilde{\pi}_1(v), \widetilde{\pi}_2(v)). \end{aligned}$$

Applying the Hölder inequality we get, for every $(u,\varphi)\in(W^{1,p}_{\Gamma_1}(\Omega)\setminus\{0\})\times W^{1,p}_{\Gamma_1}(\Omega)$ and $(v,\psi)\in(W^{1,q}_{\Gamma_2}(\Omega)\setminus\{0\})\times W^{1,q}_{\Gamma_2}(\Omega)$, the following estimates

$$\begin{aligned} |\pi_1'(u)(\varphi)| &\leq ||\nabla\varphi||_p,\ \ |\pi_2'(u)(\varphi)| \leq 2\frac{||\nabla\varphi||_p}{||\nabla u||_p}, \\ |\widetilde{\pi}_1'(v)(\psi)| &\leq ||\nabla\psi||_q,\ \ |\widetilde{\pi}_2'(v)(\psi)| \leq 2\frac{||\nabla\psi||_q}{||\nabla v||_q}. \end{aligned}$$

On one hand, from Lemma (2.2), there is a positive constant K such that $s_1(A_n)\geq K$ and $t_1(A_n)\geq K$, for every integer n. On the other hand, for every $(\varphi,\psi)\in W$,

$$\begin{aligned} D_1I(U_n,V_n)(\varphi) &= \varphi_n^1\partial_s\widetilde{I}(B_n) + \partial_u\widetilde{I}(B_n)(\varphi_n^2) + \varphi_n^1\partial_t\widetilde{I}(B_n) \\ &= \partial_u\widetilde{I}(B_n)(\varphi_n^2). \end{aligned}$$

where $\varphi_n^1 = \pi_1'(u_n)(\varphi)$ and $\varphi_n^2 = \pi_2'(u_n)(\varphi)$. Then the following estimates hold true: $|\varphi_n^1|\leq||\nabla\varphi||_p$ and $||\nabla\varphi_n^2||_p\leq\frac{2}{K}||\nabla\varphi||_p$. In the same manner, we get

$$\begin{aligned} D_2I(U_n,V_n)(\psi) &= \psi_n^1\partial_s\widetilde{I}(B_n) + \partial_v\widetilde{I}(B_n)(\psi_n^2) + \psi_n^1\partial_t\widetilde{I}(B_n) \\ &= \partial_v\widetilde{I}(B_n)(\psi_n^2). \end{aligned}$$

where $\psi_n^1 = \widetilde{\pi}_1'(v_n)(\psi)$ and $\psi_n^2 = \widetilde{\pi}_2'(v_n)(\psi)$, with the estimates $|\psi_n^1|\leq||\nabla\psi||_q$ and $||\nabla\psi_n^2||_q\leq\frac{2}{K}||\nabla\psi||_q$. Therefore

$$\begin{aligned} |D_1I(U_n,V_n)(\varphi)| &\leq \frac{1}{n}||\nabla\varphi_n^2||_p \\ &\leq \frac{2}{nK}||\nabla\varphi||_p \end{aligned}$$

and

$$
\begin{aligned}
|D_2 I(U_n, V_n)(\psi)| &\leq \frac{1}{n}||\nabla \psi_n^2||_q \\
&\leq \frac{2}{nK}||\nabla \psi||_q.
\end{aligned}
$$

We conclude easily that

$$\lim_{n\to+\infty} ||I'(U_n, V_n)||_* = 0,$$

where $I'(U_n, V_n)(\varphi, \psi) = D_1 I(U_n, V_n)(\varphi) + D_2 I(U_n, V_n)(\psi)$ and $|| \, ||_*$ is the norm on the dual space of W.

The arguments are similar if $(u_n, v_n) \in \mathbb{S}$ is a minimizing sequence of (3.21). Hence, the lemma is proved. □

Remark. For every $(u, v) \in \widetilde{W}$ and $(\lambda, \mu) \in \mathcal{D}$, one has $\widetilde{I}(s, u, t, v) = \widetilde{I}(s, |u|, t, |v|)$, $s_i(|u|, |v|, \lambda, \mu) = s_i(u, v, \lambda, \mu)$, $i \in \{1, 2\}$ and consequently $t_i(|u|, |v|, \lambda, \mu) = t_i(u, v, \lambda, \mu)$, $i \in \{1, 2\}$. Therefore, every minimizing sequence $(u_n, v_n) \in \mathbb{S}_p \times \mathbb{S}_q$ of (3.20) or (3.21) can be considered as a sequence satisfying $u_n \geq 0$ and $v_n \geq 0$ in Ω.

4 Positive solutions and the behaviour of their energy

Theorem 4.1 *Let $(\lambda, \mu) \in \mathcal{D}$. Then Problem (1.1) has at least two nontrivial solutions (U^i, V^i), $i \in \{1, 2\}$, such that $U^i \geq 0$ and $V^i \geq 0$ in Ω and $U^i \neq 0$, $V^i \neq 0$, for $i \in \{1, 2\}$.*

Proof. We will use the notations of the previous lemmas. Let $(\lambda, \mu) \in \mathcal{D}$ and consider a nonnegative minimizing sequence $(u_n, v_n) \in \mathbb{S}$ of (3.20). It is known from Lemma (3.1) that

$$
\begin{aligned}
\lim_{n\to+\infty} I(U_n, V_n) &= \alpha_1(\lambda, \mu), \\
\lim_{n\to+\infty} ||I'(U_n, V_n)||_* &= 0
\end{aligned}
$$

and that (U_n, V_n) is bounded in W. Passing if necessary to a subsequence, there are $U^1 \in W^{1,p}_{\Gamma_1}(\Omega)$ and $V^1 \in W^{1,q}_{\Gamma_2}(\Omega)$ such that

$$\begin{array}{rcl} U_n & \rightharpoonup & U^1 \;\text{ in }\; W^{1,p}_{\Gamma_1}(\Omega), \\ U_n & \to & U^1 \;\text{ in }\; L^{p_1}(\Omega) \;\text{ and }\; L^{\tilde{p}}(\Omega), \\ V_n & \rightharpoonup & V^1 \;\text{ in }\; W^{1,q}_{\Gamma_2}(\Omega), \\ V_n & \to & V^1 \;\text{ in }\; L^{q_1}(\Omega) \;\text{ and }\; L^{\tilde{q}}(\Omega), \end{array}$$

where $\tilde{p}$ and $\tilde{q}$ are specified in (2.15). At this stage, we use the well known inequalities: $\forall (x, y) \in \mathbb{R}^N$

$$\begin{array}{rcl} |x-y|^\gamma & \leq & C\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right)\cdot(x-y), \;\text{ if }\; \gamma \geq 2, \\ |x-y|^2 & \leq & C(|x|-|y|)^{2-\gamma}\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right)\cdot(x-y), \;\text{ if }\; \gamma < 2. \end{array}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^N$.

In the case $p \geq 2$, we obtain

$$\begin{array}{rcl} P(U_n - U^1) & \leq & C\displaystyle\int_\Omega \left(|\nabla U_n|^{p-2}\nabla U_n - |\nabla V_n|^{p-2}\nabla V_n\right)\cdot(\nabla U_n - \nabla V_n) \\ & = & C(D_1 I(U_n, V_n)(U_n - U^1) - D_1 I(U^1, V^1)(U_n - U^1) \; + \\ & & C\lambda \displaystyle\int_\Omega \left(|U_n|^{p_1-2}U_n - |U^1|^{p_1-2}U\right)(U_n - U^1) \; + \\ & & C(\alpha+1)\displaystyle\int_\Omega \left(U_n|U_n|^{\alpha-1}|V_n|^{\beta+1} - U^1|U^1|^{\alpha-1}|V^1|^{\beta+1}\right)(U_n - U^1). \end{array}$$

Since $\lim_{n\to+\infty} ||I'(U_n, V_n)||_* = 0$, (V_n) is bounded, and using the fact that $U_n \to U^1$ in $L^{p_1}(\Omega)$ and in $L^{\tilde{p}}(\Omega)$, $V_n \to V^1$ in in $L^{\tilde{q}}(\Omega)$, we conclude, by the Hölder inequality, that $P(U_n - U^1) \to 0$, as n goes to $+\infty$, which means that

$$U_n \longrightarrow U^1 \;\text{ in }\; W^{1,p}_{\Gamma_1}(\Omega).$$

In the case $p < 2$, a direct computation gives

$$\begin{array}{rcl} ||\nabla U_n - \nabla U^1||_p^2 & \leq & C\left(||\nabla U_n||_p^{2-p} + ||\nabla U^1||_p^{2-p}\right) \times \\ & & \displaystyle\int_\Omega \left(|\nabla U_n|^{p-2}\nabla U_n - |\nabla U^1|^{p-2}\nabla U^1\right)\cdot(\nabla U_n - \nabla U^1). \end{array}$$

Since $||\nabla U_n - \nabla U^1||_p$ is bounded, the same arguments used above show that $U_n \to U^1$ in $W^{1,p}_{\Gamma_1}(\Omega)$, as n goes to $+\infty$. In a similar way we get $V_n \to V^1$ in $W^{1,q}_{\Gamma_2}(\Omega)$, as n goes to $+\infty$.

Moreover, it is clear that (U^1, V^1) is a nontrivial solution of Problem (1.1) verifying $U^1 \geq 0$ and $V^1 \geq 0$ in Ω and $U^1 \neq 0$, $V^1 \neq 0$. On the other hand, there is a subsequence of (u_n, v_n), still denoted by (u_n, v_n) such that

$$\begin{aligned} U_n &:= s_1(u_n, v_n, \lambda, \mu)u_n \longrightarrow U^1 \quad \text{in } W^{1,p}_{\Gamma_1}(\Omega), \\ V_n &:= t_1(u_n, v_n, \lambda, \mu)v_n \longrightarrow V^1 \quad \text{in } W^{1,q}_{\Gamma_2}(\Omega). \end{aligned}$$

According to Lemma (2.2), let $(s_1, t_1) \in (0, +\infty)^2$ such that

$$\begin{cases} s_1(u_n, v_n, \lambda, \mu) & \longrightarrow & s_1 \text{ in } \mathbb{R}, \\ t_1(u_n, v_n, \lambda, \mu) & \longrightarrow & t_1 \text{ in } \mathbb{R}, \\ u_n & \longrightarrow & u^1 = \frac{U^1}{s_1} \text{ in } W^{1,p}_{\Gamma_1}(\Omega), \\ v_n & \longrightarrow & v^1 = \frac{V^1}{t_1} \text{ in } W^{1,q}_{\Gamma_2}(\Omega), \end{cases}$$

with $u^1 = \frac{U^1}{s^1} \in \mathbb{S}_p$, $v^1 = \frac{V^1}{t^1} \in \mathbb{S}_q$, $s_1 = s_1(u^1, v^1, \lambda, \mu)$ and $t_1 = t_1(u^1, v^1, \lambda, \mu)$. Therefore, $\partial_{ss}\widetilde{I}(s_1(u^1, v^1, \lambda, \mu), u^1, t_1(u^1, v^1, \lambda, \mu), v^1) > 0$.

Proceeding in the same manner with a nonnegative minimizing sequence $(\tilde{u}_n, \tilde{v}_n) \in \mathbb{S}$ of (3.21), we obtain a second nontrivial solution (U^2, V^2) of (1.1) verifying $U^2 \geq 0$ and $V^2 \geq 0$ in Ω and $U^2 \neq 0$, $V^2 \neq 0$.

Now, we have to show that $(U^1, V^1) \neq (U^2, V^2)$. Let $(s_2, t_2) \in (0, +\infty)^2$ such that

$$\begin{cases} s_2(\tilde{u}_n, \tilde{v}_n, \lambda, \mu) & \longrightarrow & s_2 \text{ in } \mathbb{R}, \\ t_2(\tilde{u}_n, \tilde{v}_n, \lambda, \mu) & \longrightarrow & t_2 \text{ in } \mathbb{R}, \\ \tilde{u}_n & \longrightarrow & u^2 = \frac{U^2}{s_2} \text{ in } W^{1,p}_{\Gamma_1}(\Omega), \\ \tilde{v}_n & \longrightarrow & v^2 = \frac{V^2}{t_2} \text{ in } W^{1,q}_{\Gamma_2}(\Omega), \end{cases}$$

with $u^2 = \frac{U^2}{s^2} \in \mathbb{S}_p$, $v^2 = \frac{V^2}{t^2} \in \mathbb{S}_q$, $s_2 = s_2(u^2, v^2, \lambda, \mu)$ and $t_2 = t_2(u^2, v^2, \lambda, \mu)$. Therefore, $\partial_{ss}\widetilde{I}(s_2(u^2, v^2, \lambda, \mu), u^2, t_2(u^2, v^2, \lambda, \mu), v^2) < 0$. Hence $(U^1, V^1) \neq (U^2, V^2)$, which ends the proof. □

In the sequel, for every $(\lambda, \mu) \in \mathcal{D}$, the functions (u^1, v^1) and (u^2, v^2) will be denoted by $(u^1(\lambda, \mu), v^1(\lambda, \mu))$ and $(u^2(\lambda, \mu), v^2(\lambda, \mu))$ respectively. Similarly, the solutions (U^i, V^i), $i \in \{1, 2\}$, will be denoted by $(U^i(\lambda, \mu), V^i(\lambda, \mu))$, $i \in \{1, 2\}$.

Theorem 4.2 *Let $(\lambda,\mu)\in\mathcal{D}$. Then*

$$\text{(i)}\quad I(U^1,V^1)<0 \ \ \textit{for}\ \ \lambda\in]0,\widehat{\lambda}(\mu)[,$$

$$\text{(ii)}\quad \begin{cases} I(U^2,V^2) & >0 \quad \textit{for}\ \lambda\in]0,\lambda_0(\mu)[,\\ I(U^2,V^2) & <0 \quad \textit{for}\ \lambda\in]\lambda_0(\mu),\widehat{\lambda}(\mu)[,\end{cases}$$

where

$$\lambda_0(\mu):=\frac{p_1}{r}\left(\frac{r}{p}\right)^{\frac{r-p_1}{r-p}}\widehat{\lambda}(\mu).$$

Proof. In this proof, μ will be fixed in $(-\infty,\mu_1)$, so we will omit the dependence on μ in the expressions which will follow. However, the dependece on λ will be specified. In particular, the Euler-Lagrange functional I will be denoted by I_λ.

(ii) Let (u,v) be an arbitrary element of $\widetilde{W}$. We denote

$$\widetilde{I}_\lambda(s,u,t(s),v)=\frac{s^p}{p}P(u)-\frac{\lambda}{p_1}s^{p_1}P_1(u)-\frac{s^r}{r}A(u,v),$$

and write

$$\widetilde{I}_\lambda(s,u,t(s),v)=s^{p_1}\widetilde{G}_\lambda(s,u,v),$$

where

$$\widetilde{G}_\lambda(s,u,v)=s^{p-p_1}\frac{P(u)}{p}-\lambda\frac{P_1(u)}{p_1}-s^{r-p_1}\frac{A(u,v)}{r}.$$

It follows that

$$\partial_s\widetilde{I}_\lambda(s,u,t(s),v)=p_1s^{p_1-1}\widetilde{G}_\lambda(s,u,v)+s^{p_1}\partial_s\widetilde{G}_\lambda(s,u,v),$$

with

$$\partial_s\widetilde{G}_\lambda(s,u,v)=s^{p-p_1-1}\left\{\frac{p-p_1}{p}P(u)-\frac{r-p_1}{r}s^{r-p}A(u,v)\right\}.$$

The real valued function $s\longmapsto\widetilde{G}_\lambda(s,u,v)$ is increasing on $]0,s_0(u,v)[$, decreasing on $]s_0(u,v),+\infty[$ and attains its unique maximum for $s=s_0(u,v)$, where

$$s_0(u,v)=\left(\frac{r}{p}\right)^{\frac{1}{r-p}}\overline{s}_\mu(u,v), \tag{4.22}$$

and $\overline{s}_\mu(u,v)$ is defined in (2.7). On the other hand, a direct computation gives

$$\widetilde{G}_\lambda(s_0(u,v),u,v)=\left(\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}\right)^{\frac{r-p_1}{r-p}}R(u,v)-\lambda P_1(u).$$

Similarly, $\widetilde{G}_\lambda(s_0(u,v),u,v) > 0$ (resp. $\widetilde{G}_\lambda(s_0(u,v),u,v) < 0$) if $\lambda < \lambda_0(u,v)$ (resp. $\lambda > \lambda_0(u,v)$) and $\widetilde{G}_{\lambda_0(u,v)}(s_0(u,v),u,v) = 0$, where

$$\lambda_0(u,v) = \frac{p_1}{r}\left(\frac{r}{p}\right)^{\frac{r-p_1}{r-p}} \lambda(u,v), \tag{4.23}$$

with $\lambda(u,v)$ given by (2.8). Thus, we get

$$\begin{cases} \widetilde{I}_\lambda(s_0(u,v),u,t(s_0(u,v)),v) > 0 & \text{if} \quad \lambda < \lambda_0(u,v), \\ \widetilde{I}_\lambda(s_0(u,v),u,t(s_0(u,v)),v) = 0 & \text{if} \quad \lambda = \lambda_0(u,v), \\ \widetilde{I}_\lambda(s_0(u,v),u,t(s_0(u,v)),v) < 0 & \text{if} \quad \lambda > \lambda_0(u,v). \end{cases} \tag{4.24}$$

First, since the function

$$\begin{array}{ccc}]0,1[& \longrightarrow & \mathbb{R} \\ t & \longmapsto & \frac{\ln t}{1-t} \end{array}$$

is increasing, then for every real numbers x, y such that $0 < x < y < 1$, one has

$$\ln\left[\frac{1}{x}\right] > \frac{1-x}{1-y}\ln\left[\frac{1}{y}\right] = \ln\left[\left(\frac{1}{y}\right)^{\frac{1-x}{1-y}}\right],$$

and consequently

$$0 < x\left(\frac{1}{y}\right)^{\frac{1-x}{1-y}} < 1.$$

In the particular case $x = p_1/r$ and $y = p/r$ we get

$$0 < \frac{p_1}{r}\left(\frac{r}{p}\right)^{\frac{r-p_1}{r-p}} < 1,$$

and therfore $0 < \lambda_0(u,v) < \lambda(u,v)$.

Moreover, for every $(u,v) \in \widetilde{W}$, one has $\widetilde{G}_{\lambda_0(u,v)}(s,u,v) < 0$ for $s \in]0,+\infty[\setminus\{s_0(u,v)\}$ and $\widetilde{G}_{\lambda_0(u,v)}(s_0(u,v),u,v) = 0$. Hence, the real valued function $s \longmapsto \widetilde{I}_{\lambda_0(u,v)}(s,u,t(s),v)$, $(s > 0)$, attains its unique maximum at $s = s_0(u,v)$ and we obtain the following interesting identity

$$s_2(u,v,\lambda_0(u,v),\mu) = s_0(u,v). \tag{4.25}$$

We will set

$$t_0(u,v) := t_2(u,v,\lambda_0(u,v),\mu).$$

On the other hand, it is clear that the functional $\lambda_0(u,v)$ is weakly lower semi-continuous on $\widetilde{W}$. Thus, the value

$$\widehat{\lambda}_0 := \inf_{(u,v)\in\widetilde{W}} \lambda_0(u,v) \tag{4.26}$$

is achieved on $\widetilde{W}$. Since $\lambda_0(u,v)$ is 0-homogeneous in u and v, we can assume that there is some $(u^*,v^*) \in \mathbb{S}_p \times \mathbb{S}_q$ such that $\widehat{\lambda}_0 = \lambda_0(u^*,v^*)$.
Now, let λ be such that $0 < \lambda < \widehat{\lambda}_0$. Then, for every $(u,v) \in \widetilde{W}$ one has $0 < \lambda < \lambda_0(u,v)$ and consequently $\widetilde{I}_\lambda(s_0(u,v),u,t(s_0(u,v)),v) > 0$ holds from (4.24). But, $s \longmapsto \widetilde{I}_\lambda(s,u,t(s),v)$, $(s>0)$ attains its unique maximum for $s = s_2(u,v,\lambda)$, hence $\widetilde{I}_\lambda(s_2(u,v,\lambda),u,t_2(u,v,\lambda),v) > 0$, for every $(u,v) \in \widetilde{W}$. In particular, we have

$$\widetilde{I}_\lambda(s_2(u^2(\lambda),v^2(\lambda),\lambda),u^2(\lambda),t_2(u^2(\lambda),v^2(\lambda),\lambda),v^2(\lambda)) > 0,$$

i.e. $I_\lambda(U^2(\lambda),V^2(\lambda)) > 0$.

If $\lambda = \widehat{\lambda}_0$, then

$$\begin{aligned} I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0),V^2(\widehat{\lambda}_0)) &= \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0),v^2(\widehat{\lambda}_0),\widehat{\lambda}_0),u^2(\widehat{\lambda}_0),t_2(u^2(\widehat{\lambda}_0),v^2(\widehat{\lambda}_0),\widehat{\lambda}_0),v^2(\widehat{\lambda}_0)) \\ &= \inf_{(u,v)\in\mathbb{S}_p\times\mathbb{S}_q} \widetilde{I}_{\widehat{\lambda}_0}(s_2(u,v,\widehat{\lambda}_0),u,t_2(u,v,\widehat{\lambda}_0),v) \\ &\le \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*,v^*),u^*,t_2(u^*,v^*),v^*) \\ &= \widetilde{I}_{\lambda_0(u^*,v^*)}(s_0(u^*,v^*),u^*,t_0(u^*,v^*),v^*) \\ &= 0 \end{aligned}$$

which implies that $I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0),V^2(\widehat{\lambda}_0)) \le 0$. In addition, it is known from (4.24) that

$$\begin{aligned} \widetilde{I}_{\widehat{\lambda}_0}(s_0(u,v),u,t_0(u,v),v) &\ge 0, \\ \widetilde{I}_{\widehat{\lambda}_0}(s_1(u,v,\widehat{\lambda}_0),u,t_1(u,v,\widehat{\lambda}_0),v) &< 0, \end{aligned}$$

for every $(u,v) \in \widetilde{W}$. Then

$$s_0(u,v) > s_1(u,v,\widehat{\lambda}_0), \quad \forall (u,v) \in \widetilde{W}.$$

It follows that

$$\begin{aligned} &\widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0),v^2(\widehat{\lambda}_0),\widehat{\lambda}_0),u^2(\widehat{\lambda}_0),t_2(u^2(\widehat{\lambda}_0),v^2(\widehat{\lambda}_0),\widehat{\lambda}_0),v^2(\widehat{\lambda}_0)) \ge \\ &\qquad \widetilde{I}_{\widehat{\lambda}_0}(s_0(u^1(\widehat{\lambda}_0),v^1(\widehat{\lambda}_0)),u^1(\widehat{\lambda}_0),t_0(u^1(\widehat{\lambda}_0),v^1(\widehat{\lambda}_0)),v^1(\widehat{\lambda}_0)) \ge 0. \end{aligned}$$

Hence,

$$\begin{aligned} I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0),V^2(\widehat{\lambda}_0)) &= \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0),v^2(\widehat{\lambda}_0),\widehat{\lambda}_0),u^2(\widehat{\lambda}_0),t_2(u^2(\widehat{\lambda}_0),v^2(\widehat{\lambda}_0),\widehat{\lambda}_0),v^2(\widehat{\lambda}_0)) \\ &= 0. \end{aligned}$$

Finally, assume that $\widehat{\lambda}_0 < \lambda < \widehat{\lambda}$. Since, for every $s \in]0,+\infty[$ and $(u,v) \in \widetilde{W}$, the real valued function $\lambda \longmapsto \widetilde{I}_\lambda(s,u,t(s),v)$ is decreasing, it follows that

$$\widetilde{I}_\lambda(s,u,t(s),v) < \widetilde{I}_{\widehat{\lambda}_0}(s,u,t(s),v), \text{ for every } s>0 \text{ and } (u,v)\in\widetilde{W}. \tag{4.27}$$

In addition, we have

$$\begin{aligned}\widetilde{I}_\lambda(s_2(u^2(\lambda),v^2(\lambda),\lambda),u^2(\lambda),t_2(u^2(\lambda),v^2(\lambda),\lambda),v^2(\lambda)) &=\\ \inf_{(u,v)\in\mathbb{S}_p\times\mathbb{S}_q}\widetilde{I}_\lambda(s_2(u,v,\lambda),u,t_2(u,v,\lambda),v) &\le\\ \widetilde{I}_\lambda(s_2(u^*,v^*,\lambda),u^*,t_2(u^*,v^*,\lambda),v^*) &<\\ \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*,v^*,\lambda),u^*,t_2(u^*,v^*,\lambda),v^*)&\end{aligned}$$

where the last inequality follows from (4.27). Moreover, the real valued function $s \longmapsto \widetilde{I}_{\widehat{\lambda}_0}(s,u^*,t(s),v^*)$, $(s>0)$, achieves its unique maximum at $s=s_0(u^*,v^*)$. Thus,

$$\begin{aligned}\widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*,v^*,\lambda),u^*,t_2(u^*,v^*,\lambda),v^*) &\le \widetilde{I}_{\widehat{\lambda}_0}(s_0(u^*,v^*),u^*,t_0(u^*,v^*),v^*)\\ &= \widetilde{I}_{\lambda_0(u^*,v^*)}(s_0(u^*,v^*),u^*,t_0(u^*,v^*),v^*)\\ &= 0.\end{aligned}$$

Hence $\widetilde{I}_\lambda(s_2(u^2(\lambda),v^2(\lambda),\lambda),u^2(\lambda),t_2(u^2(\lambda),v^2(\lambda),\lambda),v^2(\lambda))<0$, which ends the proof. □

The following result shows the subtle link existing between the characteristic value $\widehat{\lambda}_0$ defined by (4.26) and Problem (1.1).

Theorem 4.3 *If (u,v) is a solution of (4.26) then $(s_0(u,v)u, t_0(u,v)v)$ is a solution of the system (1.1) when $\lambda=\widehat{\lambda}_0$.*

Proof. Let (u,v) be a solution of (4.26). In order to simplify the notations, we set $U:=s_0(u,v)u$ and $V:=t_0(u,v)v$. Thus, for $\lambda=\widehat{\lambda}_0=\lambda_0(u,v)$ we have:

$$I_{\widehat{\lambda}_0,\mu}(U,V)=\frac{s_0(u,v)^p}{p}P(u)-\widehat{\lambda}_0\frac{s_0(u,v)^{p_1}}{p_1}P_1(u)-\frac{s_0(u,v)^r}{r}A(u,v)$$

and for every $\varphi\in W_0^{1,p}(\Omega)$:

$$D_1I_{\widehat{\lambda}_0,\mu}(U,V)(\varphi)=\frac{1}{p}P'(U)(\varphi)-\frac{\widehat{\lambda}_0}{p_1}P_1'(U)(\varphi)-\frac{1}{r}D_1A(U,V)(\varphi),$$

where

$$\begin{cases}P'(U)(\varphi) &= s_0(u,v)^{p-1}P'(u)(\varphi),\\ P_1'(U)(\varphi) &= s_0(u,v)^{p_1-1}P_1'(u)(\varphi),\\ D_1A(U,V)(\varphi) &= s_0(u,v)^{r-1}D_1A(u,v)(\varphi).\end{cases}$$

We calculate now,

$$\begin{aligned}
\widehat{\lambda}_0 P_1'(U)(\varphi) &= \lambda_0(u,v)s_0(u,v)^{p_1-1}P_1'(u)(\varphi)\\
&= \frac{p_1}{r}\left(\frac{r}{p}\right)^{\frac{r-p_1}{r-p}}\left(\frac{p-p_1}{r-p_1}\right)^{\frac{p-p_1}{r-p}}\frac{P(u)}{P_1(u)}\left(\frac{P(u)}{A(u,v)}\right)^{\frac{p-1}{r-p}}\\
&\times \frac{r-p}{r-p_1}\left(\frac{r}{p}\right)^{\frac{p_1-1}{r-p}}\left(\frac{p-p_1}{r-p_1}\frac{P(u)}{A((u,v)}\right)^{\frac{p_1-1}{r-p}}P_1'(u)(\varphi)\\
&= \frac{r-p}{r-p_1}\frac{p_1}{r}\frac{r}{p}\left(\frac{p-p_1}{r-p_1}\right)^{\frac{p-1}{r-p}}\left(\frac{r}{p}\right)^{\frac{p-1}{r-p}}\frac{P(u)}{P_1(u)}\frac{P(u)}{A(u,v)}^{\frac{p-1}{r-p}}P_1'(u)(\varphi)\\
&= \frac{r-p}{r-p_1}\frac{p_1}{p}\frac{P(u)}{P_1(u)}\left(\left(\frac{r}{p}\right)^{\frac{1}{r-p}}\left(\frac{p-p_1}{r-p_1}\right)^{\frac{1}{r-p}}\frac{P(u)}{A(u,v)}\right)^{p-1}P_1'(u)(\varphi)\\
&= \frac{p_1}{p}\frac{r-p}{r-p_1}P(u)s_0(u,v)^{p-1}\frac{P_1'(u)(\varphi)}{P_1(u)}.
\end{aligned}$$

In addition, one has

$$\begin{aligned}
D_1A(U,V)(\varphi) &= s_0(u,v)^{r-1}D_1(u,v)(\varphi)\\
&= \left(\frac{r}{p}\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}\right)^{\frac{p-1}{r-p}}\frac{r}{p}\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}D_1A(u,v)(\varphi)\\
&= \frac{r}{p}\frac{p-p_1}{r-p_1}P(u)s_0(u,v)^{p-1}\frac{D_1A(u,v)(\varphi)}{A(u,v)}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
D_1I_{\widehat{\lambda}_0,\mu}(U,V)(\varphi) &= \left[\frac{P'(u)(\varphi)}{P(u)}-\frac{r-p}{r-p_1}\frac{P_1'(u)(\varphi)}{P_1(u)}-\frac{p-p_1}{r-p_1}\frac{D_1A(u,v)(\varphi)}{A(u,v)}\right]\\
&\times \frac{P(u)s_0(u,v)^{p-1}}{p}\\
&= K\left(\frac{r-p_1}{r-p}\frac{P'(u)(\varphi)}{P(u)}-\frac{P_1'(u)(\varphi)}{P_1(u)}-\frac{p-p_1}{r-p}\frac{D_1A(u,v)(\varphi)}{A(u,v)}\right),
\end{aligned}$$

where $K := \frac{r-p}{r-p_1}\frac{P(u)}{p}s_0(u,v)^{p-1}$. On the other hand, a direct computation gives:

$$D_1\lambda_0(u,v)(\varphi) = \widehat{\lambda}_0\left(\frac{r-p_1}{r-p}\frac{P'(u)(\varphi)}{P(u)}-\frac{P_1'(u)(\varphi)}{P_1(u)}-\frac{p-p_1}{r-p}\frac{D_1A(u,v)(\varphi)}{A(u,v)}\right),$$

which is equal to zero by assumption. Hence $D_1I_{\widehat{\lambda}_0,\mu}(U,V)(\varphi) = 0$ since it is proportional to $D_1\lambda_0(u,v)(\varphi)$.

Moreover, for every $\psi \in W_0^{1,q}(\Omega)$, we get

$$D_2\lambda_0(u,v)(\psi) = -\frac{p-p_1}{r-p_1}\lambda_0(u,v)\frac{D_2A(u,v)(\psi)}{A(u,v)},$$

which is also equal to zero by assumption. This implies that $D_2A(u,v)(\psi) = 0$, since $\lambda_0(u,v) = \widehat{\lambda}_0 \neq 0$. Then

$$D_2 I_{\widehat{\lambda}_0,\mu}(U,V)(\psi) = -\frac{s_0(u,v)^r}{r} D_2A(u,v)(\psi) = 0.$$

which implies that $(s_0(u,v)u, t_0(u,v)v)$ is well a solution of the problem (1.1) with $\lambda = \widehat{\lambda}_0$. □

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