

Systems of Semilinear Higher Order Evolution Inequalities on the Heisenberg group

Abdallah El Hamidi * Amira Obeid

*Université de La Rochelle
Laboratoire de Mathématiques et Applications
Avenue Michel Crépeau
17042 La Rochelle
France*

Abstract

This paper is devoted to nonexistence results for solutions to the problem

$$(S_k^m) \begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, & \eta \in \mathbb{H}^N, \quad t \in]0, +\infty[, \quad 1 \leq i \leq m, \\ u_{m+1} = u_1, \end{cases}$$

where $\Delta_{\mathbb{H}}$ is the laplacian on the $(2N + 1)$ -dimensional Heisenberg group \mathbb{H}^N , $|\eta|_{\mathbb{H}}$ is the distance from η in \mathbb{H} to the origin, $m \geq 2$, $k \geq 1$, $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$, and $a_i \in L^\infty(\mathbb{H}^N \times]0, +\infty[)$, $1 \leq i \leq m$. These nonexistence results hold for $Q \equiv 2N + 2$ less than critical exponents which depend on k , p_i and γ_i , $1 \leq i \leq m$. For $k = 1$, $k = 2$ we retrieve the results, obtained by A. El Hamidi & M. Kirane [4], corresponding respectively to the parabolic, hyperbolic systems. In order to show that the obtained exponents are also valid for $m = 1$, we study the scalar case

$$(I_k) \quad \frac{\partial^k u}{\partial t^k} - \Delta_{\mathbb{H}}(au) \geq |\eta|_{\mathbb{H}}^\gamma |u|^p,$$

where $p > 1$, γ are real parameters and $a \in L^\infty(\mathbb{H}^N \times]0, +\infty[)$.

Key words: Critical exponent, higher order evolution inequalities, Heisenberg group

* Corresponding author.

Email addresses: aelhamid@univ-lr.fr (Abdallah El Hamidi),
aobeid@univ-lr.fr (Amira Obeid).

1 Introduction

In this section, we quote some background facts concerning the Heisenberg group. Let $\eta = (x, y, \tau) = (x_1, x_2, \dots, x_N, y_1, y_2, \dots, y_N, \tau) \in \mathbb{R}^{2N+1}$, with $N \geq 1$. The Heisenberg group \mathbb{H}^N , whose points will be denoted by $\eta = (x, y, \tau)$ is the set \mathbb{R}^{2N+1} endowed with the group operation \circ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(\langle x, \tilde{y} \rangle - \langle \tilde{x}, y \rangle)), \quad (1)$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in \mathbb{R}^N . The Laplacian $\Delta_{\mathbb{H}}$ over \mathbb{H}^N is obtained, from the vector fields $X_i = \partial_{x_i} + 2y_i \partial_{\tau}$ and $Y_i = \partial_{y_i} - 2x_i \partial_{\tau}$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^N (X_i^2 + Y_i^2). \quad (2)$$

Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^N \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right). \quad (3)$$

A natural group of dilatations on \mathbb{H}^N is given by

$$\delta_{\lambda}(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \quad \lambda > 0,$$

whose Jacobian determinant is λ^Q , where

$$Q = 2N + 2$$

is the homogeneous dimension of \mathbb{H}^N .

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of \mathbb{H}^N and homogeneous w.r.t. the dilatations δ_{λ} . More precisely, we have

$$\forall (\eta, \tilde{\eta}) \in \mathbb{H}^N \times \mathbb{H}^N, \quad \Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}}u)(\eta \circ \tilde{\eta})$$

and

$$\Delta_{\mathbb{H}}(u \circ \delta_{\lambda}) = \lambda^2 (\Delta_{\mathbb{H}}u) \circ \delta_{\lambda}.$$

It is natural to define a distance from η to the origin by

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \sum_{i=1}^N (x_i^2 + y_i^2)^2 \right)^{1/4}. \quad (4)$$

In their paper, Pohozaev & Véron [16] gave another proof of a result of Birindelli, Capuzzo-Dolcetta and Cutri [2] concerning the nonexistence of weak

solutions of the differential inequality

$$\Delta_{\mathbb{H}}(au) + |\eta|_{\mathbb{H}}^{\gamma} |u|^p \leq 0 \quad \text{in, } \mathbb{H}^N$$

for $\gamma > -2$, $1 < p \leq (Q + \gamma)/(Q - 2)$ and $a \in L^{\infty}(\mathbb{H}^N)$.

They then studied the problem of nonexistence of weak solutions to the system

$$\begin{cases} \Delta_{\mathbb{H}}(a_1u) + |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \leq 0, \\ \Delta_{\mathbb{H}}(a_2v) + |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \leq 0, \end{cases}$$

for $\gamma_i > -2$, $p_i > 1$ and $a_i \in L^{\infty}(\mathbb{H}^N)$, $i \in \{1, 2\}$. They showed that this system admits no solution defined in \mathbb{H}^N whenever $\gamma_i > -2$ and

$$Q \leq 2 + \min \left\{ \frac{\gamma_1 + 2}{p_1 - 1}; \frac{\gamma_2 + 2}{p_2 - 1} \right\}.$$

Recently, El Hamidi & Kirane [4] improved this result and gave the Fujita's exponent. Indeed, the authors showed that the system admits no solution defined in \mathbb{H}^N whenever

$$Q \leq 2 + \max \left\{ \frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1p_2 - 1}; \frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1p_2 - 1} \right\}$$

and verified that

$$\max \left\{ \frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1p_2 - 1}; \frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1p_2 - 1} \right\} \geq \min \left\{ \frac{\gamma_1 + 2}{p_1 - 1}; \frac{\gamma_2 + 2}{p_2 - 1} \right\}.$$

They then studied systems of m hypoelliptic, parabolic and hyperbolic semilinear inequalities.

In this paper, we generalize the results obtained in [4] to higher order evolution systems of m semilinear inequalities. We retrieve the critical exponent corresponding to the hypoelliptic case by setting formally $k = +\infty$.

For the convenience of the reader, we start with the case $m = 2$.

2 Higher Order Evolution Systems of two Semilinear Inequalities

Let us consider the higher order evolution system of two inequalities

$$(S_k^2) \begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta_{\mathbb{H}}(a_1 u) \geq |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1}, \\ \frac{\partial^k v}{\partial t^k} - \Delta_{\mathbb{H}}(a_2 v) \geq |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2}, \end{cases}$$

with the initial data

$$\begin{cases} u(\eta, 0) = u^{(0)}(\eta), & v(\eta, 0) = v^{(0)}(\eta) \quad \text{in } \mathbb{R}^{2N+1}, \\ \frac{\partial^i u}{\partial t^i}(\eta, 0) = u^{(i)}(\eta), & \frac{\partial^i v}{\partial t^i}(\eta, 0) = v^{(i)}(\eta), \quad i \in \{1, 2, \dots, k-1\}, \quad \text{in } \mathbb{R}^{2N+1}. \end{cases}$$

The product set $\mathbb{R}^{2N+1} \times \mathbb{R}^+$ will be denoted by $\mathbb{R}_+^{2N+1,1}$ and the integrals $\int_{\mathbb{R}^{2N+1}}$ and $\int_{\mathbb{R}_+^{2N+1,1}}$ by \int .

Definition 1. Let a_1 and a_2 be two bounded measurable functions in $\mathbb{R}_+^{2N+1,1}$. A weak solution (u, v) of the system (S_k^2) with initial data $(u^{(i)}, v^{(i)}) \in L_{loc}^1(\mathbb{R}^{2N+1}) \times L_{loc}^1(\mathbb{R}^{2N+1})$, $i \in \{0, 1, \dots, k-1\}$, is a pair of locally integrable functions (u, v) such that

$$\begin{cases} u \in L_{loc}^{p_2}(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_2} d\eta dt), \\ v \in L_{loc}^{p_1}(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_1} d\eta dt), \end{cases}$$

satisfying

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^{2N+1}} \left(u \left(a_1 \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi \right) d\eta dt + \\ & \sum_{i=0}^{k-1} (-1)^i \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x, 0) \frac{\partial^i \varphi}{\partial t^i}(x, 0) d\eta \leq 0 \end{aligned} \quad (5)$$

and

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^{2N+1}} \left(v \left(a_2 \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi \right) d\eta dt + \\ & \sum_{i=0}^{k-1} (-1)^i \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-i} v}{\partial t^{k-1-i}}(x, 0) \frac{\partial^i \varphi}{\partial t^i}(x, 0) d\eta \leq 0 \end{aligned} \quad (6)$$

for any nonnegative test function $\varphi \in C_c^{2,k}(\mathbb{R}_+^{2N+1,1})$.

Let the test function

$$\varphi_R(\eta, t) = \Phi^\lambda \left(\frac{t^{2k} + \tau^2 + |x|^4 + |y|^4}{R^4} \right), \quad (7)$$

where $\lambda \gg 1$, $R > 0$ and $\Phi \in \mathcal{D}([0, +\infty[)$ is the "standard cut-off function"

$$0 \leq \Phi(r) \leq 1, \quad \Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases} \quad (8)$$

Note that $\text{supp}(\varphi_R)$ is a subset of

$$\Omega_R = \{(x, y, \tau, t) \in \mathbb{H}^N \times [0, +\infty[; 0 \leq t^{2k} + \tau^2 + |x|^4 + |y|^4 \leq 2R^4\},$$

while $\text{supp}(\Delta_{\mathbb{H}}\varphi_R)$ and $\text{supp}\left(\frac{\partial^k \varphi_R}{\partial t^k}\right)$ are subsets of

$$\mathcal{C}_R = \{(x, y, \tau, t) \in \mathbb{H}^N \times [0, +\infty[; R^4 \leq t^{2k} + \tau^2 + |x|^4 + |y|^4 \leq 2R^4\} \quad (9)$$

and

$$\frac{\partial^i \varphi_R}{\partial t^i}(\eta, 0) = 0, \quad i \in \{1, 2, \dots, k-1\}.$$

Moreover, let

$$\rho = \frac{t^{2k} + \tau^2 + |x|^4 + |y|^4}{R^4}, \quad (10)$$

then

$$\begin{aligned} \Delta_{\mathbb{H}}\varphi_R(\eta, t) &= \frac{4(N+4)\Phi'(\rho)}{R^4} \lambda \Phi^{\lambda-1}(\rho) (|x|^2 + |y|^2) + \\ &\frac{16\Phi''(\rho)}{R^8} \lambda \Phi^{\lambda-1}(\rho) \left((|x|^6 + |y|^6) + \tau^2(|x|^2 + |y|^2) + 2\tau \langle x, y \rangle (|x|^2 - |y|^2) \right) + \\ &\frac{16\Phi'^2(\rho)}{R^8} \lambda(\lambda-1) \Phi^{\lambda-2}(\rho) \left((|x|^6 + |y|^6) + \frac{\tau^2}{4}(|x|^2 + |y|^2) + 2\tau \langle x, y \rangle (|x|^2 - |y|^2) \right). \end{aligned}$$

It follows that there is a positive constant $C_1 > 0$, independent of R , such that

$$\forall (\eta, t) \in \Omega_R, \quad |\Delta_{\mathbb{H}}\varphi_R(\eta, t)| \leq \frac{C_1}{R^2}. \quad (11)$$

Furthermore, there is a constant $C_2 > 0$, independent of R , such that

$$\left| \frac{\partial^k \varphi_R(\eta, t)}{\partial t^k} \right| \leq \frac{C_2}{R^2}.$$

Then we have the following nonexistence result:

Theorem 1. Assume that $u^{(k-1)}$ and $v^{(k-1)}$ belong to $L^1(\mathbb{R}^{2N+1})$ with $\int u^{(k-1)}(\eta) d\eta \geq 0$ and $\int v^{(k-1)}(\eta) d\eta \geq 0$. If

$$Q \leq Q_k^* = 2 \left(1 - \frac{1}{k}\right) + \frac{1}{p_1 p_2 - 1} \max\{(\gamma_1 + 2) + p_1(\gamma_2 + 2); p_2(\gamma_1 + 2) + (\gamma_2 + 2)\}$$

then there is no weak nontrivial solution (u, v) of the system (S_k^2) .

Proof. The proof is by contradiction. Let (u, v) be a nontrivial weak solution of (S_k^2) . Using the Hölder inequality, the relation (6) gives:

$$\begin{aligned} \int \varphi_R |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} d\eta dt + a(R) &\leq \int \left(|v| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + \|a_2\|_{\infty} |v| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta dt \\ &\leq \left(\int |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi_R \right)^{1/p_1} \left(\int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_1} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1})^{1-p'_1} \right)^{1/p'_1} \\ &\quad + \|a_2\|_{\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi_R \right)^{1/p_1} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p'_1} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1})^{1-p'_1} \right)^{1/p'_1}. \end{aligned}$$

Similarly (5) gives

$$\begin{aligned} \int \varphi_R |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} d\eta dt + b(R) &\leq \int \left(|u| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + \|a_1\|_{\infty} |u| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta dt \\ &\leq \left(\int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi_R \right)^{1/p_2} \left(\int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_2} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2})^{1-p'_2} \right)^{1/p'_2} \\ &\quad + \|a_1\|_{\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi_R \right)^{1/p_2} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p'_2} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2})^{1-p'_2} \right)^{1/p'_2}, \end{aligned}$$

where

$$a(R) = \int_{\mathbb{H}^N} v^{(k-1)}(\eta) \varphi_R(\eta, 0) d\eta$$

and

$$b(R) = \int_{\mathbb{H}^N} u^{(k-1)}(\eta) \varphi_R(\eta, 0) d\eta.$$

If we set

$$\left\{ \begin{array}{l} I(R) = \int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi_R \, d\eta \, dt, \\ J(R) = \int |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi_R \, d\eta \, dt, \\ \mathcal{A}_{p_i, \gamma_i}(R) = \int |\Delta_{\mathbb{H}} \varphi_R|^{p'_i} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i})^{1-p'_i} \, d\eta \, dt, \quad i \in \{1, 2\}, \\ \mathcal{B}_{p_i, \gamma_i}(R) = \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_i} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i})^{1-p'_i} \, d\eta \, dt, \quad i \in \{1, 2\}, \end{array} \right.$$

we then have the following system of inequalities

$$\left\{ \begin{array}{l} I(R) + a(R) \leq C J^{1/p_1}(R) \left[(\mathcal{A}_{p_1, \gamma_1}(R))^{1/p'_1} + (\mathcal{B}_{p_1, \gamma_1}(R))^{1/p'_1} \right], \\ J(R) + b(R) \leq C I^{1/p_2}(R) \left[(\mathcal{A}_{p_2, \gamma_2}(R))^{1/p'_2} + (\mathcal{B}_{p_2, \gamma_2}(R))^{1/p'_2} \right], \end{array} \right. \quad (12)$$

where C is a positive constant independent of R .

Note that if λ is selected sufficiently large then the integrals $\mathcal{A}_{p_i, \gamma_i}(R)$ and $\mathcal{B}_{p_i, \gamma_i}(R)$, $i \in \{1, 2\}$, are convergent. Indeed, the exponent of φ_R in the integrands of $\mathcal{A}_{p_i, \gamma_i}(R)$ and $\mathcal{B}_{p_i, \gamma_i}(R)$ is positive if λ is selected large enough.

Moreover, the system (12) implies that neither u nor v is trivial. Indeed, if v is trivial then $J(R) = 0$ and we have $I(R) + a(R) \leq 0$. Since $a(R)$ is uniformly bounded w.r.t. R , it follows that $I(R)$ is also uniformly bounded w.r.t. R . Using the fact that $I(R)$ is increasing in R , the monotone convergence theorem shows that the function $u \in L^{p_2}(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_2} \, d\eta \, dt)$. Whence, we have

$$\lim_{R \rightarrow +\infty} (I(R) + a(R)) = \int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \, d\eta \, dt + \int v^{(k-1)}(\eta) \, d\eta \leq 0,$$

and the function u is then trivial, which is impossible.

Now, let ε be a real number such that $0 < \varepsilon < 1$, there is $R_1 > 0$ such that $I(R_1) > 0$. Since

$$0 \leq \lim_{R \rightarrow +\infty} a(R) < +\infty,$$

there exists $R_2 \geq R_1$ such that $-\varepsilon I(R_1) \leq a(R)$, for any $R \geq R_2$. Moreover, the function $I(R)$ is nonnegative and increasing of R , then for any $R \geq R_2$, the inequalities

$$I(R) + a(R) \geq I(R) - \varepsilon I(R_1) \geq (1 - \varepsilon)I(R)$$

hold true. The same arguments imply that there is $R_3 \geq R_2$ such that $J(R) + b(R) \geq (1 - \varepsilon)J(R)$ for any $R \geq R_3$. Finally, the system (12) gives

$$\begin{cases} I(R) \leq \frac{C}{1-\varepsilon} J^{\frac{1}{p_1}}(R) \left[(\mathcal{A}_{p_1, \gamma_1}(R))^{\frac{1}{p_1}} + (\mathcal{B}_{p_1, \gamma_1}(R))^{\frac{1}{p_1}} \right], \\ J(R) \leq \frac{C}{1-\varepsilon} I^{\frac{1}{p_2}}(R) \left[(\mathcal{A}_{p_2, \gamma_2}(R))^{\frac{1}{p_2}} + (\mathcal{B}_{p_2, \gamma_2}(R))^{\frac{1}{p_2}} \right], \end{cases} \quad (13)$$

for any $R \geq R_3$. Then, there is a constant $C > 0$, independent of R , such that

$$\begin{cases} I(R)^{1-\frac{1}{p_1 p_2}} \leq C \left[\mathcal{A}_{p_1, \gamma_1}^{\frac{1}{p_1}} + \mathcal{B}_{p_1, \gamma_1}^{\frac{1}{p_1}} \right] \left[\mathcal{A}_{p_2, \gamma_2}^{\frac{1}{p_2}} + \mathcal{B}_{p_2, \gamma_2}^{\frac{1}{p_2}} \right]^{\frac{1}{p_1}}, \\ J(R)^{1-\frac{1}{p_1 p_2}} \leq C \left[\mathcal{A}_{p_1, \gamma_1}^{\frac{1}{p_1}} + \mathcal{B}_{p_1, \gamma_1}^{\frac{1}{p_1}} \right]^{\frac{1}{p_2}} \left[\mathcal{A}_{p_2, \gamma_2}^{\frac{1}{p_2}} + \mathcal{B}_{p_2, \gamma_2}^{\frac{1}{p_2}} \right]. \end{cases} \quad (14)$$

In order to estimate the integrals $\mathcal{A}_{p_i, \gamma_i}(R)$ and $\mathcal{B}_{p_i, \gamma_i}(R)$, $i \in \{1, 2\}$, we introduce the scaled variables

$$\begin{cases} \tilde{t} = R^{-\frac{2}{k}} t \\ \tilde{\tau} = R^{-2} \tau, \\ \tilde{x} = R^{-1} x, \\ \tilde{y} = R^{-1} y. \end{cases} \quad (15)$$

Using the fact that $\text{supp} \varphi_R \subset \Omega_R$, we conclude that

$$\begin{cases} \mathcal{A}_{p_i, \gamma_i}(R) \leq C R^{2N+2+2/k-2p'_i+\gamma_i(1-p'_i)}, & i \in \{1, 2\}, \\ \mathcal{B}_{p_i, \gamma_i}(R) \leq C R^{2N+2+2/k-2p'_i+\gamma_i(1-p'_i)}, & i \in \{1, 2\}, \end{cases}$$

which is equivalent to

$$\begin{cases} \mathcal{A}_{p_i, \gamma_i}(R) \leq C R^{Q+2/k-2p'_i+\gamma_i(1-p'_i)}, & i \in \{1, 2\}, \\ \mathcal{B}_{p_i, \gamma_i}(R) \leq C R^{Q+2/k-2p'_i+\gamma_i(1-p'_i)}, & i \in \{1, 2\}. \end{cases}$$

Consequently, the estimates

$$I(R)^{1-\frac{1}{p_1 p_2}} \leq C R^{\sigma_I} \quad \text{and} \quad J(R)^{1-\frac{1}{p_1 p_2}} \leq C R^{\sigma_J}$$

hold true, where

$$\sigma_I = \frac{1}{p_1} \left((Q + 2/k) \frac{p_2 - 1}{p_2} - 2 - \frac{\gamma_2}{p_2} \right) + \left((Q + 2/k) \frac{p_1 - 1}{p_1} - 2 - \frac{\gamma_1}{p_1} \right)$$

and

$$\sigma_J = \frac{1}{p_2} \left((Q + 2/k) \frac{p_1 - 1}{p_1} - 2 - \frac{\gamma_1}{p_1} \right) + \left((Q + 2/k) \frac{p_2 - 1}{p_2} - 2 - \frac{\gamma_2}{p_2} \right).$$

Finally, the exponents σ_I or σ_J are less than zero if, and only if,

$$\begin{aligned} Q \leq Q_k^* &= 2 + \frac{1}{p_1 p_2 - 1} \max\{(\gamma_1 + 2) + p_1(\gamma_2 + 2); p_2(\gamma_1 + 2) + (\gamma_2 + 2)\} - 2/k \\ &= 2(1 - 1/k) + \frac{1}{p_1 p_2 - 1} \max\{(\gamma_1 + 2) + p_1(\gamma_2 + 2); p_2(\gamma_1 + 2) + (\gamma_2 + 2)\}. \end{aligned}$$

In this case, the integrals $I(R)$ and $J(R)$, which are increasing in R , are bounded uniformly w.r.t. R . Using the monotone convergence theorem, we deduce that

$$(u, v) \in L^{p_2} \left(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_2} d\eta dt \right) \times L^{p_1} \left(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_1} d\eta dt \right).$$

Note that instead of (12) we have more precisely

$$\begin{cases} I(R) + a(R) \leq C \tilde{J}^{1/p_1}(R) \left[(\mathcal{A}_{p_1, \gamma_1}(R))^{1/p_1'} + (\mathcal{B}_{p_1, \gamma_1}(R))^{1/p_1'} \right] \\ J(R) + b(R) \leq C \tilde{I}^{1/p_2}(R) \left[(\mathcal{A}_{p_2, \gamma_2}(R))^{1/p_2'} + (\mathcal{B}_{p_2, \gamma_2}(R))^{1/p_2'} \right] \end{cases} \quad (16)$$

where

$$\tilde{I}(R) = \int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi_R d\eta dt$$

and

$$\tilde{J}(R) = \int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi_R d\eta dt,$$

where \mathcal{C}_R is defined in (9). Finally, using the dominated convergence theorem, we obtain

$$\lim_{R \rightarrow +\infty} \tilde{I}(R) = \lim_{R \rightarrow +\infty} \tilde{J}(R) = 0.$$

Hence,

$$\lim_{R \rightarrow +\infty} (I(R) + a(R)) = \int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} d\eta dt + \int v^{(k-1)}(\eta) d\eta = 0$$

and

$$\lim_{R \rightarrow +\infty} (J(R) + b(R)) = \int |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} d\eta dt + \int u^{(k-1)}(\eta) d\eta = 0,$$

which implies that $u \equiv v \equiv 0$. This completes the proof. \square

Corollary 1. *Assume that $\int u^{(k-1)}(\eta) d\eta \geq 0$, $\int v^{(k-1)}(\eta) d\eta \geq 0$, and*

$$Q \leq Q_k^* = 2(1 - 1/k) + \max\{X_1, X_2\},$$

where the vector $(X_1, X_2)^T$ is the solution of the linear system

$$\begin{pmatrix} -1 & p_1 \\ p_2 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \end{pmatrix}. \quad (17)$$

Then there is no weak nontrivial solution (u, v) of the system (S_k^2) .

Proof. The vector $(X_1, X_2)^T$ is given by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -1 & p_1 \\ p_2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \end{pmatrix} = \frac{1}{p_1 p_2 - 1} \begin{pmatrix} (\gamma_1 + 2) + p_1(\gamma_2 + 2) \\ p_2(\gamma_1 + 2) + (\gamma_2 + 2) \end{pmatrix}.$$

\square

Remark 1. *To determine the critical exponent Q_e^* corresponding to the hypoelliptic system*

$$\begin{cases} -\Delta_{\mathbb{H}}(a_1 u) \geq |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1}, \\ -\Delta_{\mathbb{H}}(a_2 v) \geq |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2}, \end{cases}$$

it suffices to tend formally k to infinity in the exponent Q_k^* and obtain

$$Q_e^* = 2 + \max\{X_1, X_2\}.$$

Now, we are able to treat the case of systems of m semilinear inequalities.

3 Higher-Order Evolution Systems of m Semilinear Inequalities

Let (X_1, X_2, \dots, X_m) be the solution of the linear system

$$\begin{pmatrix} -1 & p_1 & 0 & \dots & 0 \\ 0 & -1 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & p_{m-1} \\ p_m & 0 & \dots & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{pmatrix} = \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \\ \vdots \\ \gamma_{m-1} + 2 \\ \gamma_m + 2 \end{pmatrix}, \quad (18)$$

where $p_i > 1$ and γ_i are given real numbers, $i \in \{1, 2, \dots, m\}$.

Consider the system

$$(S_k^m) \begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_i+1} |u_{i+1}|^{p_i+1}, & \eta \in \mathbb{R}^{2N+1}, \quad t \in]0, +\infty[, \quad 1 \leq i \leq m, \\ u_{m+1} = u_1, \end{cases}$$

where $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$, and the initial data $(u_i^{(0)}, u_i^{(1)}, \dots, u_i^{(k-1)}) \in [L_{loc}^1(\mathbb{R}^{2N+1})]^k$, $1 \leq i \leq m$.

Definition 2. Let a_i , $i \in \{1, 2, \dots, m\}$, be m bounded measurable functions on $\mathbb{R}_+^{2N+1,1}$. A weak solution (u_1, \dots, u_m) of the system (S_k^m) on $\mathbb{R}_+^{2N+1,1}$ is a vector of locally integrable functions (u_1, \dots, u_m) such that

$$u_i \in L_{loc}^{p_i}(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_i} d\eta dt), \quad i \in \{1, 2, \dots, m\},$$

satisfying

$$\int_0^\infty \int_{\mathbb{R}^{2N+1}} \left(u_i \left(a_i \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{\mathbb{H}}^{\gamma_i+1} |u_{i+1}|^{p_i+1} \varphi \right) d\eta dt + \sum_{j=0}^{k-1} (-1)^j \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-j} u_i}{\partial t^{k-1-j}}(\eta, 0) \frac{\partial^j \varphi}{\partial t^j}(\eta, 0) d\eta \leq 0, \quad i \in \{1, 2, \dots, m-1\}, \quad (19)$$

and

$$\int_0^\infty \int_{\mathbb{R}^{2N+1}} \left(u_m \left(a_m \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi \right) d\eta dt + \sum_{j=0}^{k-1} (-1)^j \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-j} u_m}{\partial t^{k-1-j}}(\eta, 0) \frac{\partial^j \varphi}{\partial t^j}(\eta, 0) d\eta \leq 0 \quad (20)$$

for any nonnegative test function $\varphi \in C_c^{2,k}(\mathbb{R}_+^{2N+1,1})$.

Theorem 2. Assume that $u_i^{(k-1)} \in L^1(\mathbb{R}^{2N+1})$, $1 \leq i \leq m$, and

$$\int_{\mathbb{R}^{2N+1}} u_i^{(k-1)}(\eta) d\eta \geq 0, \quad 1 \leq i \leq m.$$

Then, $Q \leq 2(1 - 1/k) + \max\{X_1, X_2, \dots, X_m\}$ implies that the system (S_k^m) has no nontrivial solution.

Proof. In order to simplify the proof, we treat only the case $m = 3$, the general case can be established in the same manner.

Let (u_1, u_2, u_3) be a nontrivial weak solution of (S_k^m) . The inequalities (19) and (20), with $\varphi = \varphi_R$ defined by (7), imply that

$$\begin{aligned} \int \varphi_R |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} d\eta dt + a(R) &\leq \int \left(|u_3| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + \|a_3\|_{\infty} |u_3| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta dt \\ &\leq \left(\int |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} \varphi_R \right)^{1/p_3} \left(\int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_3} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_3})^{1-p'_3} \right)^{1/p'_3} \\ &\quad + \|a_3\|_{\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} \varphi_R \right)^{1/p_3} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p'_3} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_3})^{1-p'_3} \right)^{1/p'_3}, \end{aligned}$$

$$\begin{aligned} \int \varphi_R |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} d\eta dt + b(R) &\leq \int \left(|u_1| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + \|a_1\|_{\infty} |u_1| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta dt \\ &\leq \left(\int |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi_R \right)^{1/p_1} \left(\int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_1} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1})^{1-p'_1} \right)^{1/p'_1} \\ &\quad + \|a_1\|_{\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi_R \right)^{1/p_1} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p'_1} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1})^{1-p'_1} \right)^{1/p'_1} \end{aligned}$$

and

$$\begin{aligned} \int \varphi_R |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} d\eta dt + c(R) &\leq \int \left(|u_2| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + \|a_2\|_{\infty} |u_2| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta dt \\ &\leq \left(\int |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} \varphi_R \right)^{1/p_2} \left(\int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_2} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2})^{1-p'_2} \right)^{1/p'_2} \\ &\quad + \|a_2\|_{\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} \varphi_R \right)^{1/p_2} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p'_2} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2})^{1-p'_2} \right)^{1/p'_2}. \end{aligned}$$

Let

$$\left\{ \begin{array}{l} I_i(R) = \int |\eta|_{\mathbb{H}}^{\gamma_i} |u_i|^{p_i} \varphi_R d\eta, \quad 1 \leq i \leq 3, \\ \mathcal{A}_i(R) = \int |\Delta_{\mathbb{H}} \varphi_R|^{p'_i} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i})^{1-p'_i}, \quad 1 \leq i \leq 3, \\ \mathcal{B}_i(R) = \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_i} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i})^{1-p'_i} d\eta dt, \quad 1 \leq i \leq 3. \end{array} \right.$$

Then there is a positive constant C such that

$$\left\{ \begin{array}{l} I_1 \leq C I_3^{1/p_3} (\mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'}), \\ I_2 \leq C I_1^{1/p_1} (\mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'}), \\ I_3 \leq C I_2^{1/p_2} (\mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'}). \end{array} \right.$$

Whence, the estimates

$$\left\{ \begin{array}{l} I_1^{1-\frac{1}{p_1 p_2 p_3}} \leq C (\mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'})^{\frac{1}{p_2 p_3}} (\mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'})^{\frac{1}{p_3}} (\mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'}), \\ I_2^{1-\frac{1}{p_1 p_2 p_3}} \leq C (\mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'}) (\mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'})^{\frac{1}{p_1 p_3}} (\mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'})^{\frac{1}{p_1}}, \\ I_3^{1-\frac{1}{p_1 p_2 p_3}} \leq C (\mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'})^{\frac{1}{p_2}} (\mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'}) (\mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'})^{\frac{1}{p_1 p_2}}, \end{array} \right.$$

hold true.

In order to estimate the expressions I_i , $1 \leq i \leq 3$, we use the scaled variables (15) and obtain

$$I_i^{1-\frac{1}{p_1 p_2 p_3}} \leq C R^{\sigma_i}, \quad 1 \leq i \leq 3,$$

where

$$\left\{ \begin{array}{l} \sigma_1 = \left(1 - \frac{1}{p_1 p_2 p_3}\right) \left(Q - 2 + \frac{2}{k} - \frac{(\gamma_1+2)+p_1(\gamma_2+2)+p_1 p_2(\gamma_3+2)}{p_1 p_2 p_3 - 1}\right), \\ \sigma_2 = \left(1 - \frac{1}{p_1 p_2 p_3}\right) \left(Q - 2 + \frac{2}{k} - \frac{p_2 p_3(\gamma_1+2)+(\gamma_2+2)+p_2(\gamma_3+2)}{p_1 p_2 p_3 - 1}\right), \\ \sigma_3 = \left(1 - \frac{1}{p_1 p_2 p_3}\right) \left(Q - 2 + \frac{2}{k} - \frac{p_3(\gamma_1+2)+p_1 p_3(\gamma_2+2)+(\gamma_3+2)}{p_1 p_2 p_3 - 1}\right). \end{array} \right.$$

Now, we require that, at least, one of σ_i , $1 \leq i \leq 3$, is less than zero, which is equivalent to $Q \leq 2(1 - 1/k) + \max\{X_1, X_2, X_3\}$, where the vector $(X_1, X_2, X_3)^T$ is the solution of

$$\begin{pmatrix} -1 & p_1 & 0 \\ 0 & -1 & p_2 \\ p_3 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \\ \gamma_3 + 2 \end{pmatrix}. \quad (21)$$

Following the arguments used in the proof of Theorem 1, we conclude that $(u_1, u_2, u_3) \equiv (0, 0, 0)$. This ends the proof by contradiction. \square

Remark 2. *To determine the critical exponent Q_e^* corresponding to the hypoelliptic system*

$$(S_k^m) \begin{cases} -\Delta(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_i+1} |u_{i+1}|^{p_i+1}, & x \in \mathbb{R}^{2N+1}, \quad 1 \leq i \leq m, \\ u_{m+1} = u_1, \end{cases}$$

it suffices to tend formally k to infinity in the exponent Q_k^* and obtain

$$Q_e^* = 2 + \max\{X_1, X_2, \dots, X_m\}.$$

In the following section, we show that the result of Theorem 2 is also valid for $m = 1$.

4 Higher Order Evolution Semilinear Inequalities

Let us consider the higher inequality (I_k) with the initial data

$$\begin{cases} u(\eta, 0) = u^{(0)}(\eta), & \text{in } \mathbb{R}^{2N+1}, \\ \frac{\partial^i u}{\partial t^i}(\eta, 0) = u^{(i)}(\eta), \quad i \in \{1, 2, \dots, k-1\}, & \text{in } \mathbb{R}^{2N+1}. \end{cases}$$

Definition 3. *Let a a bounded measurable functions in $\mathbb{R}_+^{2N+1,1}$. A weak solution u of the inequality (I_k) with initial data $u^{(i)} \in L_{loc}^1(\mathbb{R}^{2N+1})$, $i \in \{0, 1, \dots, k-1\}$, is a locally integrable function u such that*

$$u \in L_{loc}^p(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma} d\eta dt),$$

satisfying

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^{2N+1}} \left(u \left(a \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|^\gamma |u|^p \varphi \right) d\eta dt + \\ & \sum_{i=0}^{k-1} (-1)^i \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(\eta, 0) \frac{\partial^i \varphi}{\partial t^i}(\eta, 0) d\eta \leq 0, \end{aligned} \quad (22)$$

for any nonnegative test function $\varphi \in C_c^{2,k}(\mathbb{R}_+^{2N+1,1})$.

Theorem 3. Assume that $u^{(k-1)} \in L^1(\mathbb{R}^{2N+1})$ and $\int u^{(k-1)}(\eta) d\eta \geq 0$. If

$$Q \leq 2 \left(1 - \frac{1}{k} \right) + \frac{\gamma + 2}{p - 1},$$

then there is no weak nontrivial solution u of the system (I_k) .

Proof. Let u be a nontrivial weak solution of (I_k) . Using the Hölder inequality, the equation (22) gives:

$$\begin{aligned} & \int \varphi_R |\eta|^\gamma |u|^p d\eta dt + \tilde{a}(R) \leq \int \left(|u| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + |u| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta dt \\ & \leq \left(\int |\eta|^\gamma |u|^p \varphi_R dx dt \right)^{1/p} \left(\int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'} (\varphi_R |\eta|^\gamma)^{1-p'} d\eta dt \right)^{1/p'} \\ & + \|a\|_\infty \left(\int |\eta|^\gamma |u|^p \varphi_R d\eta dt \right)^{1/p} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p'} (\varphi_R |\eta|^\gamma)^{1-p'} d\eta dt \right)^{1/p'}, \end{aligned}$$

where

$$\tilde{a}(R) = \int_{\mathbb{H}^N} u^{(k-1)}(\eta) \varphi_R(\eta, 0) d\eta.$$

Let us set

$$\begin{aligned} \tilde{I}(R) &= \int \varphi_R |\eta|^\gamma |u|^p d\eta dt, \\ \tilde{\mathcal{A}}_{p,\gamma}(R) &= \int |\Delta_{\mathbb{H}} \varphi_R|^{p'} (\varphi_R |\eta|^\gamma)^{1-p'} d\eta dt, \end{aligned}$$

and

$$\tilde{\mathcal{B}}_{p,\gamma}(R) = \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'} (\varphi_R |\eta|^\gamma)^{1-p'} d\eta dt.$$

Following the same method described in the last proof, we obtain

$$I(R) \leq C \left(\tilde{\mathcal{A}}_{p,\gamma}(R)^{1/p'} + \tilde{\mathcal{B}}_{p,\gamma}(R)^{1/p'} \right) I(R)^{1/p}, \quad (23)$$

where C is a positive constant independent of R . Using the same scaled variables as before, we have the estimate

$$I(R)^{1-1/p} \leq CR^\sigma,$$

where

$$\sigma = -2 - \frac{\gamma}{p} + \left(Q + \frac{2}{k}\right) \frac{p-1}{p}.$$

Now, we require $\sigma \leq 0$ which is equivalent to

$$Q \leq 2 \left(1 - \frac{1}{k}\right) + \frac{\gamma+2}{p-1}. \quad (24)$$

In this case, the integral $I(R)$, increasing in R , is bounded uniformly w.r.t. R . The monotone convergence theorem implies that $|\eta|_{\mathbb{H}}^{\gamma}|u|^p$ belongs to $L^1(\mathbb{R}_+^{2N+1,1})$. Note that instead of (23) we have more precisely

$$\begin{aligned} \int |\eta|_{\mathbb{H}}^{\gamma}|u|^p \varphi_R \, d\eta \, dt &\leq \|a\|_{L^\infty} \left(\int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma}|u|^p \varphi_R \, d\eta \, dt \right)^{1/p} \left(\tilde{\mathcal{A}}_{p,\gamma}(R)^{1/p'} + \tilde{\mathcal{B}}_{p,\gamma}(R)^{1/p'} \right) \\ &\leq C \int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma}|u|^p \varphi_R \, d\eta \, dt, \end{aligned}$$

where \mathcal{C}_R is defined in (9). Finally, using the dominated convergence theorem, we obtain that

$$\lim_{R \rightarrow +\infty} \int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma}|u|^p \varphi_R \, d\eta \, dt = 0.$$

Hence

$$\int |\eta|_{\mathbb{H}}^{\gamma}|u|^p \, d\eta \, dt = 0,$$

which implies that $u \equiv 0$. This contradicts the fact that u is a nontrivial weak solution of (I_k) , which achieves the proof. \square

Remark 3. *To determine the critical exponent for the hypoelliptic inequality*

$$-\Delta_{\mathbb{H}}(au) \geq |\eta|_{\mathbb{H}}^{\gamma}|u|^p,$$

it suffices to tend formally k to infinity and obtain $2 + \frac{\gamma+2}{p-1}$.

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