Systems of Semilinear Higher Order Evolution Inequalities on the Heisenberg group

Abdallah El Hamidi * Amira Obied

Université de La Rochelle
Laboratoire de Mathématiques et Applications
Avenue Michel Crépeau
17042 La Rochelle
France

Abstract

This paper is devoted to nonexistence results for solutions to the problem

\[ (S^m_k) \]
\[ \frac{\partial^k u}{\partial t^k} - \Delta_H(a_i u_i) \geq |\eta|_H^{\gamma_{i+1}}|u_{i+1}|^{p_{i+1}}, \quad \eta \in \mathbb{H}^N, \quad t \in ]0, +\infty[, \quad 1 \leq i \leq m, \]
\[ u_{m+1} = u_1, \]

where \( \Delta_H \) is the laplacian on the \((2N+1)\)–dimensional Heisenberg group \( \mathbb{H}^N \), \( |\eta|_H \) is the distance from \( \eta \) in \( \mathbb{H} \) to the origin, \( m \geq 2, \quad k \geq 1, \quad p_{m+1} = p_1, \quad \gamma_{m+1} = \gamma_1, \) and \( a_i \in L^\infty(\mathbb{H}^N \times ]0, +\infty[), \quad 1 \leq i \leq m. \) These nonexistence results hold for \( Q \equiv 2N + 2 \) less than critical exponents which depend on \( k, \ p_i \) and \( \gamma_i, \ 1 \leq i \leq m. \) For \( k = 1, \ k = 2 \) we retrieve the results, obtained by A. El Hamidi & M. Kirane [4], corresponding respectively to the parabolic, hyperbolic systems. In order to show that the obtained exponents are also valid for \( m = 1, \) we study the scalar case

\[ (I_k) \]
\[ \frac{\partial^k u}{\partial t^k} - \Delta_H(au) \geq |\eta|_H^\gamma |u|^p, \]

where \( p > 1, \ \gamma \) are real parameters and \( a \in L^\infty(\mathbb{H}^N \times ]0, +\infty[). \)

Key words: Critical exponent, higher order evolution inequalities, Heisenberg group

* Corresponding author.

Email addresses: aelhamid@univ-lr.fr (Abdallah El Hamidi), aobeid@univ-lr.fr (Amira Obied).
1 Introduction

In this section, we quote some background facts concerning the Heisenberg group. Let \( \eta = (x, y, \tau) = (x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N, \tau) \in \mathbb{R}^{2N+1} \), with \( N \geq 1 \). The Heisenberg group \( \mathbb{H}^N \), whose points will be denoted by \( \eta = (x, y, \tau) \) is the set \( \mathbb{R}^{2N+1} \) endowed with the group operation \( \circ \) defined by

\[
\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(<x, \tilde{y}> - <\tilde{x}, y>)),
\]

where \(<, >\) is the usual inner product in \( \mathbb{R}^N \). The Laplacian \( \Delta_{\mathbb{H}} \) over \( \mathbb{H}^N \) is obtained, from the vector fields \( X_i = \partial_{x_i} + 2y_i \partial_{\tau} \) and \( Y_i = \partial_{y_i} - 2x_i \partial_{\tau} \), by

\[
\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2).
\]

Explicit computation gives the expression

\[
\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).
\]

A natural group of dilatations on \( \mathbb{H}^N \) is given by

\[
\delta_{\lambda}(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \quad \lambda > 0,
\]

whose Jacobian determinant is \( \lambda^Q \), where

\[
Q = 2N + 2
\]

is the homogeneous dimension of \( \mathbb{H}^N \).

The operator \( \Delta_{\mathbb{H}} \) is a degenerate elliptic operator. It is invariant with respect to the left translation of \( \mathbb{H}^N \) and homogeneous w.r.t. the dilatations \( \delta_{\lambda} \). More precisely, we have

\[
\forall (\eta, \tilde{\eta}) \in \mathbb{H}^N \times \mathbb{H}^N, \quad \Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}} u)(\eta \circ \tilde{\eta})
\]

and

\[
\Delta_{\mathbb{H}}(u \circ \delta_{\lambda}) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_{\lambda}.
\]

It is natural to define a distance from \( \eta \) to the origin by

\[
|\eta|_H = \left( \tau^2 + \sum_{i=1}^{N} (x_i^2 + y_i^2)^2 \right)^{1/4}.
\]

In their paper, Pohozaev & Véron [16] gave another proof of a result of Birindelli, Capuzzo-Dolcetta and Cutri [2] concerning the nonexistence of weak
solutions of the differential inequality

\[ \Delta_H(au) + |\eta|_{H^1}^\gamma |u|^p \leq 0 \text{ in } \mathbb{H}^N \]

for \( \gamma > -2, 1 < p \leq (Q + \gamma)/(Q - 2) \) and \( a \in L^\infty(\mathbb{H}^N) \).

They then studied the problem of nonexistence of weak solutions to the system

\[
\begin{aligned}
\Delta_H(a_1 u) + |\eta|_{H^1}^\gamma_1 |v|^{p_1} &\leq 0, \\
\Delta_H(a_2 v) + |\eta|_{H^1}^\gamma_2 |u|^{p_2} &\leq 0,
\end{aligned}
\]

for \( \gamma_i > -2, p_i > 1 \) and \( a_i \in L^\infty(\mathbb{H}^N), i \in \{1, 2\} \). They showed that this system admits no solution defined in \( \mathbb{H}^N \) whenever \( \gamma_i > -2 \) and

\[ Q \leq 2 + \min \left\{ \frac{\gamma_1 + 2}{p_1 - 1}; \frac{\gamma_2 + 2}{p_2 - 1} \right\} . \]

Recently, El Hamidi & Kirane [4] improved this result and gave the Fujita’s exponent. Indeed, the authors showed that the system admits no solution defined in \( \mathbb{H}^N \) whenever

\[ Q \leq 2 + \max \left\{ \frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1 p_2 - 1}; \frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1 p_2 - 1} \right\} . \]

and verified that

\[ \max \left\{ \frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1 p_2 - 1}; \frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1 p_2 - 1} \right\} \geq \min \left\{ \frac{\gamma_1 + 2}{p_1 - 1}; \frac{\gamma_2 + 2}{p_2 - 1} \right\} . \]

They then studied systems of \( m \) hypoelliptic, parabolic and hyperbolic semilinear inequalities.

In this paper, we generalize the results obtained in [4] to higher order evolution systems of \( m \) semilinear inequalities. We retrieve the critical exponent corresponding to the hypoelliptic case by setting formally \( k = +\infty \).

For the convenience of the reader, we start with the case \( m = 2 \).
2 Higher Order Evolution Systems of Two Semilinear Inequalities

Let us consider the higher order evolution system of two inequalities

\[
\begin{align*}
\begin{cases}
\pfrac{\partial^k u}{\partial t^k} - \Delta_H(a_1 u) & \geq |\eta|_{H}^{\gamma_1} |\nu|^{p_1}, \\
\pfrac{\partial^k v}{\partial t^k} - \Delta_H(a_2 v) & \geq |\eta|_{H}^{\gamma_2} |u|^{p_2},
\end{cases}
\end{align*}
\]

with the initial data

\[
\begin{align*}
\begin{cases}
u(\eta, 0) = u^{(0)}(\eta), & v(\eta, 0) = v^{(0)}(\eta) \quad \text{in } \mathbb{R}^{2N+1}, \\
pfrac{\partial^i u}{\partial t^i}(\eta, 0) = u^{(i)}(\eta), & pfrac{\partial^i v}{\partial t^i}(\eta, 0) = v^{(i)}(\eta), \quad i \in \{1, 2, \ldots, k-1\} \quad \text{in } \mathbb{R}^{2N+1}.
\end{cases}
\end{align*}
\]

The product set \( \mathbb{R}^{2N+1} \times \mathbb{R}^+ \) will be denoted by \( \mathbb{R}^{2N+1,1}_+ \) and the integrals \( \int_{\mathbb{R}^{2N+1}} \) and \( \int_{\mathbb{R}^{2N+1,1}} \) by \( \mathring{\int} \).

**Definition 1.** Let \( a_1 \) and \( a_2 \) be two bounded measurable functions in \( \mathbb{R}^{2N+1,1}_+ \).

A weak solution \((u, v)\) of the system \((S^2_k)\) with initial data \((u^{(i)}, v^{(i)}) \in L^1_{\text{loc}}(\mathbb{R}^{2N+1}) \times L^1_{\text{loc}}(\mathbb{R}^{2N+1}) , \ i \in \{0, 1, \ldots, k-1\} , \) is a pair of locally integrable functions \((u, v)\) such that

\[
\begin{align*}
\begin{cases}
u \in L^p_{\text{loc}}(\mathbb{R}^{2N+1,1}_+ , |\eta|_{H}^{\gamma_1} \, d\eta \, dt), \\
v \in L^{p_1}_{\text{loc}}(\mathbb{R}^{2N+1,1}_+ , |\eta|_{H}^{\gamma_2} \, d\eta \, dt),
\end{cases}
\end{align*}
\]

satisfying

\[
\begin{align*}
\mathring{\int}_0^\infty \int_{\mathbb{R}^{2N+1}} & \left( u \left( a_1 \Delta_H \varphi - (-1)^k \pfrac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{H}^{\gamma_1} |\nu|^{p_1} \varphi \right) \, d\eta \, dt + \\
\sum_{i=0}^{k-1} & (-1)^i \int_{\mathbb{R}^{2N+1}} \pfrac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x, 0) \pfrac{\partial^i \varphi}{\partial t^i}(x, 0) \, d\eta \leq 0
\end{align*}
\]

and

\[
\begin{align*}
\mathring{\int}_0^\infty \int_{\mathbb{R}^{2N+1}} & \left( v \left( a_2 \Delta_H \varphi - (-1)^k \pfrac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{H}^{\gamma_2} |u|^{p_2} \varphi \right) \, d\eta \, dt + \\
\sum_{i=0}^{k-1} & (-1)^i \int_{\mathbb{R}^{2N+1}} \pfrac{\partial^{k-1-i} v}{\partial t^{k-1-i}}(x, 0) \pfrac{\partial^i \varphi}{\partial t^i}(x, 0) \, d\eta \leq 0
\end{align*}
\]

for any nonnegative test function \( \varphi \in C^2_{c}(\mathbb{R}^{2N+1,1}) \).
Let the test function
\[ \varphi_R(\eta, t) = \Phi^\lambda \left( \frac{t^{2k} + \tau^2 + |x|^4 + |y|^4}{R^4} \right), \quad (7) \]
where \( \lambda \gg 1 \), \( R > 0 \), and \( \Phi \in \mathcal{D}([0, +\infty]) \) is the "standard cut-off function"
\[ 0 \leq \Phi(r) \leq 1, \quad \Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases} \quad (8) \]

Note that \( \text{supp}(\varphi_R) \) is a subset of
\[ \Omega_R = \{ (x, y, \tau, t) \in \mathbb{H}^N \times [0, +\infty] ; \ 0 \leq t^{2k} + \tau^2 + |x|^4 + |y|^4 \leq 2R^4 \}, \]
while \( \text{supp}(\Delta_{\mathbb{H}}\varphi_R) \) and \( \text{supp} \left( \frac{\partial^k \varphi_R}{\partial t^k} \right) \) are subsets of
\[ C_R = \{ (x, y, \tau, t) \in \mathbb{H}^N \times [0, +\infty] ; \ R^4 \leq t^{2k} + \tau^2 + |x|^4 + |y|^4 \leq 2R^4 \} \quad (9) \]
and
\[ \frac{\partial^k \varphi_R}{\partial t^k}(\eta, 0) = 0, \quad i \in \{1, 2, \ldots, k-1\}. \]

Moreover, let
\[ \rho = \frac{t^{2k} + \tau^2 + |x|^4 + |y|^4}{R^4}, \quad (10) \]
then
\[ \Delta_{\mathbb{H}}\varphi_R(\eta, t) = \frac{4(N + 4)\Phi'(\rho)}{R^4} \lambda \Phi^{\lambda-1}(\rho)(|x|^2 + |y|^2) + \]
\[ 16\Phi''(\rho) \lambda \Phi^{\lambda-1}(\rho) \left( (|x|^6 + |y|^6) + \tau^2(|x|^2 + |y|^2) + 2\tau < x, y > (|x|^2 - |y|^2) \right) + \]
\[ \frac{16 \Phi'^2(\rho)}{R^8} \lambda (\lambda - 1) \Phi^{\lambda-2}(\rho) \left( (|x|^6 + |y|^6) + \frac{\tau^2}{4}(|x|^2 + |y|^2) + 2\tau < x, y > (|x|^2 - |y|^2) \right). \]

It follows that there is a positive constant \( C_1 > 0 \), independent of \( R \), such that
\[ \forall (\eta, t) \in \Omega_R, \quad |\Delta_{\mathbb{H}}\varphi_R(\eta, t)| \leq \frac{C_1}{R^2}. \quad (11) \]
Furthermore, there is a constant \( C_2 > 0 \), independent of \( R \), such that
\[ \left| \frac{\partial^k \varphi_R(\eta, t)}{\partial t^k} \right| \leq \frac{C_2}{R^2}. \]

Then we have the following nonexistence result:
Theorem 1. Assume that \( u^{(k-1)} \) and \( v^{(k-1)} \) belong to \( L^1(\mathbb{R}^{2N+1}) \) with \( \int u^{(k-1)}(\eta) \, d\eta \geq 0 \) and \( \int v^{(k-1)}(\eta) \, d\eta \geq 0 \). If

\[
Q \leq Q^*_k = 2 \left( 1 - \frac{1}{k} \right) + \frac{1}{p_1 p_2 - 1} \max \{ (\gamma_1 + 2) + p_1 (\gamma_2 + 2) : p_2 (\gamma_1 + 2) + (\gamma_2 + 2) \}
\]

then there is no weak nontrivial solution \((u, v)\) of the system \((S^2_k)\).

Proof. The proof is by contradiction. Let \((u, v)\) be a nontrivial weak solution of \((S^2_k)\). Using the Hölder inequality, the relation (6) gives:

\[
\int \varphi_R |\eta|^{\gamma_2} |u|^{p_2} \, d\eta \, dt + a(R) \leq \int \left( |v| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + ||a_2||_\infty |v| |\Delta \varphi_R| \right) \, d\eta \, dt \\
\leq \left( \int |\eta|^{\gamma_2} |v|^{p_1} \varphi_R \right)^{1/p_1} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|_{\varphi_R} \left( \varphi_R |\eta|^{\gamma_2} \right)^{1-p_1} \right)^{1/p'_1} \\
+ ||a_2||_\infty \left( \int |\eta|^{\gamma_2} |v|^{p_1} \varphi_R \right)^{1/p_1} \left( \int |\Delta \varphi_R| \varphi_R \left( \varphi_R |\eta|^{\gamma_2} \right)^{1-p_1} \right)^{1/p'_1}.
\]

Similarly (5) gives

\[
\int \varphi_R |\eta|^{\gamma_1} |v|^{p_1} \, d\eta \, dt + b(R) \leq \int \left( |u| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + ||a_1||_\infty |u| |\Delta \varphi_R| \right) \, d\eta \, dt \\
\leq \left( \int |\eta|^{\gamma_1} |u|^{p_2} \varphi_R \right)^{1/p_2} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|_{\varphi_R} \left( \varphi_R |\eta|^{\gamma_2} \right)^{1-p_2} \right)^{1/p'_2} \\
+ ||a_1||_\infty \left( \int |\eta|^{\gamma_1} |u|^{p_2} \varphi_R \right)^{1/p_2} \left( \int |\Delta \varphi_R| \varphi_R \left( \varphi_R |\eta|^{\gamma_2} \right)^{1-p_2} \right)^{1/p'_2},
\]

where

\[
a(R) = \int_{\mathbb{H}^N} u^{(k-1)}(\eta) \varphi_R(\eta, 0) \, d\eta
\]

and

\[
b(R) = \int_{\mathbb{H}^N} v^{(k-1)}(\eta) \varphi_R(\eta, 0) \, d\eta.
\]
If we set
\[
\begin{align*}
I(R) &= \int |\eta|^{\gamma_2} |u|^{p_2} \varphi_R \, d\eta \, dt, \\
J(R) &= \int |\eta|^{\gamma_1} |v|^{p_1} \varphi_R \, d\eta \, dt, \\
A_{p_i,\gamma_i}(R) &= \int |\Delta \varphi_R|^k |(\varphi_R |\eta|^{\gamma_i})^{1-p_i'} \, d\eta \, dt, \quad i \in \{1, 2\}, \\
B_{p_i,\gamma_i}(R) &= \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^k |(\varphi_R |\eta|^{\gamma_i})^{1-p_i'} \, d\eta \, dt, \quad i \in \{1, 2\},
\end{align*}
\]
we then have the following system of inequalities
\[
\begin{align*}
I(R) + a(R) &\leq C J^{1/p_1}(R) \left[ (A_{p_1,\gamma_1}(R))^{1/p_1'} + (B_{p_1,\gamma_1}(R))^{1/p_1'} \right], \\
J(R) + b(R) &\leq C I^{1/p_2}(R) \left[ (A_{p_2,\gamma_2}(R))^{1/p_2'} + (B_{p_2,\gamma_2}(R))^{1/p_2'} \right],
\end{align*}
\]  
(12)

where \( C \) is a positive constant independent of \( R \).

Note that if \( \lambda \) is selected sufficiently large then the integrals \( A_{p_i,\gamma_i}(R) \) and \( B_{p_i,\gamma_i}(R), \quad i \in \{1, 2\}, \) are convergent. Indeed, the exponent of \( \varphi_R \) in the integrands of \( A_{p_i,\gamma_i}(R) \) and \( B_{p_i,\gamma_i}(R) \) is positive if \( \lambda \) is selected large enough.

Moreover, the system (12) implies that neither \( u \) nor \( v \) is trivial. Indeed, if \( v \) is trivial then \( J(R) = 0 \) and we have \( I(R) + a(R) \leq 0 \). Since \( a(R) \) is uniformly bounded w.r.t. \( R \), it follows that \( I(R) \) is also uniformly bounded w.r.t. \( R \).

Using the fact that \( I(R) \) is increasing in \( R \), the monotone convergence theorem shows that the function \( u \in L^{p_2} \left( \mathbb{R}^{2N+1,1} \times |\eta|^{\gamma_2} \right. \left. d\eta \, dt \right) \). Whence, we have
\[
\lim_{R \to +\infty} (I(R) + a(R)) = \int |\eta|^{\gamma_2} |u|^{p_2} \, d\eta \, dt + \int v^{(k-1)}(\eta) \, d\eta \leq 0,
\]
and the function \( u \) is then trivial, which is impossible.

Now, let \( \varepsilon \) be a real number such that \( 0 < \varepsilon < 1 \), there is \( R_1 > 0 \) such that \( I(R_1) > 0 \). Since
\[
0 \leq \lim_{R \to +\infty} a(R) < +\infty,
\]
there exists \( R_2 \geq R_1 \) such that \( -\varepsilon I(R_1) \leq a(R) \), for any \( R \geq R_2 \). Moreover, the function \( I(R) \) is nonnegative and increasing of \( R \), then for any \( R \geq R_2 \), the inequalities
\[
I(R) + a(R) \geq I(R) - \varepsilon I(R_1) \geq (1 - \varepsilon) I(R)
\]
hold true. The same arguments imply that there is $R_3 \geq R_2$ such that $J(R) + b(R) \geq (1 - \varepsilon) J(R)$ for any $R \geq R_3$. Finally, the system (12) gives

$$
\begin{align*}
I(R) &\leq \frac{C}{\varepsilon R} J_{p_1}^\frac{1}{p_1} (R) \left[ (A_{p_1,\gamma_1}(R))^{\frac{1}{p_1}} + (B_{p_1,\gamma_1}(R))^{\frac{1}{p_1}} \right], \\
J(R) &\leq \frac{C}{\varepsilon R} J_{p_2}^\frac{1}{p_2} (R) \left[ (A_{p_2,\gamma_2}(R))^{\frac{1}{p_2}} + (B_{p_2,\gamma_2}(R))^{\frac{1}{p_2}} \right],
\end{align*}
$$

for any $R \geq R_3$. Then, there is a constant $C > 0$, independent of $R$, such that

$$
\begin{align*}
I(R)^{1 - \frac{1}{\gamma_1 p_1}} &\leq C \left[ A_{p_1,\gamma_1}^{\frac{1}{p_1}} + B_{p_1,\gamma_1}^{\frac{1}{p_1}} \right] \left[ A_{p_2,\gamma_2}^{\frac{1}{p_2}} + B_{p_2,\gamma_2}^{\frac{1}{p_2}} \right]^{\frac{1}{p_1}}, \\
J(R)^{1 - \frac{1}{\gamma_1 p_1}} &\leq C \left[ A_{p_1,\gamma_1}^{\frac{1}{p_1}} + B_{p_1,\gamma_1}^{\frac{1}{p_1}} \right]^{\frac{1}{\gamma_2}} \left[ A_{p_2,\gamma_2}^{\frac{1}{p_2}} + B_{p_2,\gamma_2}^{\frac{1}{p_2}} \right].
\end{align*}
$$

In order to estimate the integrals $A_{p_i,\gamma_i}(R)$ and $B_{p_i,\gamma_i}(R)$, $i \in \{1, 2\}$, we introduce the scaled variables

$$
\begin{align*}
\tilde{t} &= R^{-\frac{2}{p_1}} t, \\
\tilde{\tau} &= R^{-2} \tau, \\
\tilde{x} &= R^{-1} x, \\
\tilde{y} &= R^{-1} y.
\end{align*}
$$

Using the fact that $\text{supp} \varphi_R \subset \Omega_R$, we conclude that

$$
\begin{align*}
A_{p_i,\gamma_i}(R) &\leq C R^{2N + 2 - 2p_i' + \gamma_i (1 - p_i')} , \quad i \in \{1, 2\}, \\
B_{p_i,\gamma_i}(R) &\leq C R^{2N + 2 - 2p_i' + \gamma_i (1 - p_i')} , \quad i \in \{1, 2\},
\end{align*}
$$

which is equivalent to

$$
\begin{align*}
A_{p_i,\gamma_i}(R) &\leq C R^{Q + 2 - 2p_i' + \gamma_i (1 - p_i')} , \quad i \in \{1, 2\}, \\
B_{p_i,\gamma_i}(R) &\leq C R^{Q + 2 - 2p_i' + \gamma_i (1 - p_i')} , \quad i \in \{1, 2\}.
\end{align*}
$$

Consequently, the estimates

$$
I(R)^{1 - \frac{1}{\gamma_1 p_1}} \leq C R^{\gamma_1} \quad \text{and} \quad J(R)^{1 - \frac{1}{\gamma_1 p_1}} \leq C R^{\gamma_1}
$$
hold true, where
\[
\sigma_I = \frac{1}{p_1} \left( (Q + 2/k) \frac{p_2 - 1}{p_2} - 2 - \frac{\gamma_2}{p_2} \right) + \left( (Q + 2/k) \frac{p_1 - 1}{p_1} - 2 - \frac{\gamma_1}{p_1} \right)
\]
and
\[
\sigma_J = \frac{1}{p_2} \left( (Q + 2/k) \frac{p_1 - 1}{p_1} - 2 - \frac{\gamma_1}{p_1} \right) + \left( (Q + 2/k) \frac{p_2 - 1}{p_2} - 2 - \frac{\gamma_2}{p_2} \right).
\]

Finally, the exponents \( \sigma_I \) or \( \sigma_J \) are less than zero if, and only if,
\[
Q \leq Q_k^* = 2 + \frac{1}{p_1 p_2 - 1} \max\{(\gamma_1 + 2) + p_1(\gamma_2 + 2); p_2(\gamma_1 + 2) + (\gamma_2 + 2)\} - \frac{2}{k} = 2(1 - 1/k) + \frac{1}{p_1 p_2 - 1} \max\{(\gamma_1 + 2) + p_1(\gamma_2 + 2); p_2(\gamma_1 + 2) + (\gamma_2 + 2)\}.
\]

In this case, the integrals \( I(R) \) and \( J(R) \), which are increasing in \( R \), are bounded uniformly w.r.t. \( R \). Using the monotone convergence theorem, we deduce that
\[
(u, v) \in L^{p_2} \left( \mathbb{R}^{2N+1,1}_+, |\eta|^{\gamma_2}_H d\eta dt \right) \times L^{p_1} \left( \mathbb{R}^{2N+1,1}_+, |\eta|^{\gamma_1}_H d\eta dt \right).
\]

Note that instead of (12) we have more precisely
\[
\begin{align*}
\begin{cases}
I(R) + a(R) &\leq C \tilde{J}^{1/p_1}(R) \left[ (A_{p_1, \gamma_1}(R))^{1/p_1'} + (B_{p_1, \gamma_1}(R))^{1/p_1'} \right] \\
J(R) + b(R) &\leq C \tilde{I}^{1/p_2}(R) \left[ (A_{p_2, \gamma_2}(R))^{1/p_2'} + (B_{p_2, \gamma_2}(R))^{1/p_2'} \right]
\end{cases}
\end{align*}
\]
(16)

where
\[
\tilde{I}(R) = \int_{C_R} |\eta|^{\gamma_2}_H |u|^{p_2} \varphi_R d\eta dt
\]
and
\[
\tilde{J}(R) = \int_{C_R} |\eta|^{\gamma_1}_H |v|^{p_1} \varphi_R d\eta dt,
\]
where \( C_R \) is defined in (9). Finally, using the dominated convergence theorem, we obtain
\[
\lim_{R \to +\infty} \tilde{I}(R) = \lim_{R \to +\infty} \tilde{J}(R) = 0.
\]

Hence,
\[
\lim_{R \to +\infty} (I(R) + a(R)) = \int |\eta|^{\gamma_2}_H |u|^{p_2} d\eta dt + \int v^{(k-1)}(\eta) d\eta = 0
\]
and
\[
\lim_{R \to +\infty} (J(R) + b(R)) = \int |\eta|^{\gamma_1}_H |v|^{p_1} d\eta dt + \int u^{(k-1)}(\eta) d\eta = 0.
\]
which implies that $u \equiv v \equiv 0$. This completes the proof. \hfill \Box

**Corollary 1.** Assume that $\int u^{(k-1)}(\eta) \, d\eta \geq 0$, $\int v^{(k-1)}(\eta) \, d\eta \geq 0$, and

$$Q \leq Q_k^* = 2(1 - 1/k) + \max\{X_1, X_2\},$$

where the vector $(X_1, X_2)^T$ is the solution of the linear system

$$\begin{pmatrix} -1 & p_1 \\ p_2 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \end{pmatrix}. \tag{17}$$

Then there is no weak nontrivial solution $(u, v)$ of the system $(S_k^2)$.

**Proof.** The vector $(X_1, X_2)^T$ is given by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -1 & p_1 \\ p_2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \end{pmatrix} = \frac{1}{p_1 p_2 - 1} \begin{pmatrix} (\gamma_1 + 2) + p_1 (\gamma_2 + 2) \\ p_2 (\gamma_1 + 2) + (\gamma_2 + 2) \end{pmatrix}. \hfill \Box$$

**Remark 1.** To determine the critical exponent $Q^*_e$ corresponding to the hypoelliptic system

$$\begin{align*}
-\Delta_{\mathbb{R}^n}(a_1 u) &\geq |\eta|^{\gamma_1} |v|^{p_1}, \\
-\Delta_{\mathbb{R}^n}(a_2 v) &\geq |\eta|^{\gamma_2} |u|^{p_2},
\end{align*}$$

it suffices to tend formally $k$ to infinity in the exponent $Q_k^*$ and obtain

$$Q^*_e = 2 + \max\{X_1, X_2\}.$$ 

Now, we are able to treat the case of systems of $m$ semilinear inequalities.
Let \((X_1, X_2, ..., X_m)\) be the solution of the linear system
\[
\begin{pmatrix}
-1 & p_1 & 0 & \ldots & 0 \\
0 & -1 & p_2 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & p_{m-1} \\
p_m & 0 & \ldots & 0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_{m-1} \\
X_m \\
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 + 2 \\
\gamma_2 + 2 \\
\vdots \\
\gamma_{m-1} + 2 \\
\gamma_m + 2 \\
\end{pmatrix},
\tag{18}
\]
where \(p_i > 1\) and \(\gamma_i\) are given real numbers, \(i \in \{1, 2, ..., m\}\).

Consider the system
\[
(S_k^m)
\begin{align*}
\frac{\partial^ku_i}{\partial t^k} - \Delta_{\mathbb{H}}(a_i u_i) &\geq |\eta|^{\gamma_i+1}|u_{i+1}|^{p_{i+1}} , \quad \eta \in \mathbb{R}^{2N+1}, \ t \in [0, +\infty[ , \ 1 \leq i \leq m, \\
u_{m+1} = u_1,
\end{align*}
\]
where \(p_{m+1} = p_1, \gamma_{m+1} = \gamma_1,\) and the initial data \((u_i(0), u_i(1), ..., u_i(k-1)) \in \left[L^1_{\text{loc}}(\mathbb{R}^{2N+1})\right]^k, \ 1 \leq i \leq m.\)

**Definition 2.** Let \(a_i, \ i \in \{1, 2, ..., m\},\) be \(m\) bounded measurable functions on \(\mathbb{R}^{2N+1}_+.\) A weak solution \((u_1, ..., u_m)\) of the system \((S_k^m)\) on \(\mathbb{R}^{2N+1}_+\) is a vector of locally integrable functions \((u_1, ..., u_m)\) such that
\[
u_i \in L^p_{\text{loc}}(\mathbb{R}^{2N+1}_+, |\eta|^{\gamma_i} \, d\eta \, dt), \quad i \in \{1, 2, ..., m\},
\]
satisfying
\[
\int_0^\infty \int_{\mathbb{R}^{2N+1}_+} \left( u_i \left( a_i \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^{k} \varphi}{\partial t^k} \right) + |\eta|^{\gamma_i+1}|u_{i+1}|^{p_{i+1}} \varphi \right) \, d\eta \, dt + \\
\sum_{j=0}^{k-1} (-1)^j \int_{\mathbb{R}^{2N+1}_+} \frac{\partial^{k-1-j} u_i}{\partial t^{k-1-j}} (\eta, 0) \frac{\partial^{j} \varphi}{\partial t^{j}} (\eta, 0) \, d\eta \leq 0, \quad i \in \{1, 2, ..., m-1\}, \tag{19}
\]
and
\[
\int_0^\infty \int_{\mathbb{R}^{2N+1}_+} \left( u_m \left( a_m \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^{k} \varphi}{\partial t^k} \right) + |\eta|^{\gamma_1}|u_{1}|^{p_{1}} \varphi \right) \, d\eta \, dt + \\
\sum_{j=0}^{k-1} (-1)^j \int_{\mathbb{R}^{2N+1}_+} \frac{\partial^{k-1-j} u_m}{\partial t^{k-1-j}} (\eta, 0) \frac{\partial^{j} \varphi}{\partial t^{j}} (\eta, 0) \, d\eta \leq 0 \tag{20}
\]
for any nonnegative test function \(\varphi \in C^2_c(\mathbb{R}^{2N+1}_+).\)
Theorem 2. Assume that $u_i^{(k-1)} \in L^1(\mathbb{R}^{2N+1})$, $1 \leq i \leq m$, and

$$
\int_{\mathbb{R}^{2N+1}} u_i^{(k-1)}(\eta) \, d\eta \geq 0, \quad 1 \leq i \leq m.
$$

Then, $Q \leq 2(1 - 1/k) + \max\{X_1, X_2, \ldots, X_m\}$ implies that the system $(S^m_k)$ has no nontrivial solution.

Proof. In order to simplify the proof, we treat only the case $m = 3$, the general case can be established in the same manner.

Let $(u_1, u_2, u_3)$ be a nontrivial weak solution of $(S^m_k)$. The inequalities (19) and (20), with $\varphi = \varphi_R$ defined by (7), imply that

$$
\int \varphi_R |\eta|^{\gamma_3} |u_1|^{p_1} \, d\eta \, dt + a(R) \leq \int \left( |u_3| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + ||a_3||_{\infty} |u_3| |\Delta H \varphi_R| \right) \, d\eta \, dt
$$

$$
\leq \left( \int |\eta|^{\gamma_3} |u_3|^{p_3} \varphi_R \right)^{1/p_3} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_3} \varphi_R \right)^{1/p'_3} (\varphi_R |\eta|^{\gamma_3} - \rho_3)^{1/p'_3},
$$

$$
\int \varphi_R |\eta|^{\gamma_2} |u_2|^{p_2} \, d\eta \, dt + b(R) \leq \int \left( |u_1| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + ||a_1||_{\infty} |u_1| |\Delta H \varphi_R| \right) \, d\eta \, dt
$$

$$
\leq \left( \int |\eta|^{\gamma_2} |u_1|^{p_1} \varphi_R \right)^{1/p_1} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_1} \varphi_R \right)^{1/p'_1} (\varphi_R |\eta|^{\gamma_2} - \rho_1)^{1/p'_1},
$$

and

$$
\int \varphi_R |\eta|^{\gamma_3} |u_3|^{p_3} \, d\eta \, dt + c(R) \leq \int \left( |u_2| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + ||a_2||_{\infty} |u_2| |\Delta H \varphi_R| \right) \, d\eta \, dt
$$

$$
\leq \left( \int |\eta|^{\gamma_3} |u_2|^{p_2} \varphi_R \right)^{1/p_2} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'_2} \varphi_R \right)^{1/p'_2} (\varphi_R |\eta|^{\gamma_3} - \rho_2)^{1/p'_2}.
$$
Let
\[
I_i(R) = \int |\eta|^2 |u_i|^p \varphi_R \, d\eta, \quad 1 \leq i \leq 3,
\]
\[
A_i(R) = \int |\Delta \varphi_R|^p' (\varphi_R |\eta|^\gamma)^{1-p'} \, \, d\eta, \quad 1 \leq i \leq 3,
\]
\[
B_i(R) = \int |\frac{\partial^k \varphi_R}{\partial t^k}|^p' (\varphi_R |\eta|^\gamma)^{1-p'} \, \, d\eta \, dt, \quad 1 \leq i \leq 3.
\]
Then there is a positive constant \( C \) such that
\[
\begin{align*}
I_1 & \leq C I_3^{1/p_3} \left( A_3^{1/p_3} + B_3^{1/p_3} \right), \\
I_2 & \leq C I_1^{1/p_1} \left( A_1^{1/p_1} + B_1^{1/p_1} \right), \\
I_3 & \leq C I_2^{1/p_2} \left( A_2^{1/p_2} + B_2^{1/p_2} \right).
\end{align*}
\]
Whence, the estimates
\[
\begin{align*}
I_1^{1-\frac{1}{p_1p_2p_3}} & \leq C \left( A_1^{1/p_1} + B_1^{1/p_1} \right)^{\frac{1}{p_1p_2}} \left( A_2^{1/p_2} + B_2^{1/p_2} \right)^{\frac{1}{p_2p_3}} \left( A_3^{1/p_3} + B_3^{1/p_3} \right)^{\frac{1}{p_1}}, \\
I_2^{1-\frac{1}{p_1p_2p_3}} & \leq C \left( A_1^{1/p_1} + B_1^{1/p_1} \right) \left( A_2^{1/p_2} + B_2^{1/p_2} \right)^{\frac{1}{p_1p_3}} \left( A_3^{1/p_3} + B_3^{1/p_3} \right)^{\frac{1}{p_1}}, \\
I_3^{1-\frac{1}{p_1p_2p_3}} & \leq C \left( A_1^{1/p_1} + B_1^{1/p_1} \right)^{\frac{1}{p_2}} \left( A_2^{1/p_2} + B_2^{1/p_2} \right) \left( A_3^{1/p_3} + B_3^{1/p_3} \right)^{\frac{1}{p_1p_2}},
\end{align*}
\]
hold true.

In order to estimate the expressions \( I_i, 1 \leq i \leq 3 \), we use the scaled variables (15) and obtain
\[
I_i^{1-\frac{1}{p_1p_2p_3}} \leq C R^{\sigma_i}, \quad 1 \leq i \leq 3,
\]
where
\[
\begin{align*}
\sigma_1 & = \left( 1 - \frac{1}{p_1p_2p_3} \right) \left( Q - 2 + \frac{2}{k} - \frac{\gamma_1+2+p_1(\gamma_2+2)+p_1p_2(\gamma_3+2)}{p_1p_2p_3-1} \right), \\
\sigma_2 & = \left( 1 - \frac{1}{p_1p_2p_3} \right) \left( Q - 2 + \frac{2}{k} - \frac{p_2p_3(\gamma_1+2)+\gamma_2+2+p_2(\gamma_3+2)}{p_1p_2p_3-1} \right), \\
\sigma_3 & = \left( 1 - \frac{1}{p_1p_2p_3} \right) \left( Q - 2 + \frac{2}{k} - \frac{p_3(\gamma_1+2)+p_1p_3(\gamma_2+2)+\gamma_3+2}{p_1p_2p_3-1} \right).
\end{align*}
\]
Now, we require that, at least, one of $\sigma_i$, $1 \leq i \leq 3$, is less than zero, which is equivalent to $Q \leq 2(1 - 1/k) + \max\{X_1, X_2, X_3\}$, where the vector $(X_1, X_2, X_3)^T$ is the solution of

$$
\begin{pmatrix}
-1 & p_1 & 0 \\
0 & -1 & p_2 \\
p_3 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix}
= 
\begin{pmatrix}
\gamma_1 + 2 \\
\gamma_2 + 2 \\
\gamma_3 + 2
\end{pmatrix}.
\tag{21}
$$

Following the arguments used in the proof of Theorem 1, we conclude that $(u_1, u_2, u_3) \equiv (0, 0, 0)$. This ends the proof by contradiction. \hfill \Box

**Remark 2.** To determine the critical exponent $Q^*_e$ corresponding to the hypoelliptic system

$$
\begin{cases}
-\Delta(a_i u_i) \geq |\eta|^{\gamma_i+1} |u_{i+1}|^{p_i+1}, & x \in \mathbb{R}^{2N+1}, \quad 1 \leq i \leq m, \\
u_{m+1} = u_1,
\end{cases}
$$

it suffices to tend formally $k$ to infinity in the exponent $Q^*_e$ and obtain

$$
Q^*_e = 2 + \max\{X_1, X_2, ..., X_m\}.
$$

In the following section, we show that the result of Theorem 2 is also valid for $m = 1$.

### 4 Higher Order Evolution Semilinear Inequalities

Let us consider the higher inequality $(I_k)$ with the initial data

$$
\begin{cases}
u(\eta, 0) = u^{(0)}(\eta), & \text{in } \mathbb{R}^{2N+1}, \\
\frac{\partial^i \nu}{\partial t^i}(\eta, 0) = u^{(i)}(\eta), & \text{in } \mathbb{R}^{2N+1}.
\end{cases}
$$

**Definition 3.** Let $a$ a bounded measurable functions in $\mathbb{R}_+^{2N+1,1}$. A weak solution $u$ of the inequality $(I_k)$ with initial data $u^{(i)} \in L^1_{\text{loc}}(\mathbb{R}^{2N+1})$, $i \in \{0, 1, ..., k - 1\}$, is a locally integrable function $u$ such that

$$
u \in L^p_{\text{loc}}(\mathbb{R}_+^{2N+1,1}; |\eta|^{\gamma} \, d\eta dt),
$$
satisfying

\[
\int_0^\infty \int_{\mathbb{R}^{2N+1}} \left( u \left( a \Delta_H \varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_H^2 |u|_F \right) \, d\eta \, dt +
\]

\[
\sum_{i=0}^{k-1} (-1)^i \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(\eta, 0) \frac{\partial^i \varphi}{\partial t^i}(\eta, 0) \, d\eta \leq 0,
\]  
(22)

for any nonnegative test function \( \varphi \in C^2_c(\mathbb{R}^{2N+1}) \).

**Theorem 3.** Assume that \( u^{(k-1)} \in L^1(\mathbb{R}^{2N+1}) \) and \( \int u^{(k-1)}(\eta) \, d\eta \geq 0 \). If

\[
Q \leq 2 \left( 1 - \frac{1}{k} \right) + \frac{\gamma + 2}{p - 1},
\]

then there is no weak nontrivial solution \( u \) of the system \((I_k)\).

**Proof.** Let \( u \) be a nontrivial weak solution of \((I_k)\). Using the Hölder inequality, the equation (22) gives:

\[
\int \varphi_R |\eta|_H^2 |u|_F \, d\eta \, dt + \tilde{a}(R) \leq \int \left( |u| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + |u|_F \Delta_H \varphi_R \right) \, d\eta \, dt
\]

\[
\leq \left( \int |\eta|_H^2 |u|^p \varphi_R \, dx \, dt \right)^{1/p} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'} (\varphi_R |\eta|_H^2)^{1-p'} \, d\eta \, dt \right)^{1/p'}
\]

\[
+ ||u||_\infty \left( \int |\eta|_H^2 |u|^p \varphi_R \, d\eta \, dt \right)^{1/p} \left( \int |\Delta_H \varphi_R|^{p'} (\varphi_R |\eta|_H^2)^{1-p'} \, d\eta \, dt \right)^{1/p'}
\]

where

\[
\tilde{a}(R) = \int_{\mathbb{R}^N} u^{(k-1)}(\eta) \varphi_R(\eta, 0) \, d\eta.
\]

Let us set

\[
\tilde{I}(R) = \int \varphi_R |\eta|_H^2 |u|_F \, d\eta \, dt,
\]

\[
\tilde{A}_{p,\gamma}(R) = \int |\Delta_H \varphi_R|^{p'} (\varphi_R |\eta|_H^2)^{1-p'} \, d\eta \, dt,
\]

and

\[
\tilde{B}_{p,\gamma}(R) = \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'} (\varphi_R |\eta|_H^2)^{1-p'} \, d\eta \, dt.
\]

Following the same method described in the last proof, we obtain

\[
I(R) \leq C \left( \tilde{A}_{p,\gamma}(R)^{1/p'} + \tilde{B}_{p,\gamma}(R)^{1/p'} \right) I(R)^{1/p},
\]

(23)

where \( C \) is a positive constant independent of \( R \). Using the same scaled variables as before, we have the estimate

\[
I(R)^{1-1/p} \leq CR^2,
\]

15
where
\[ \sigma = -2 - \frac{\gamma}{p} + \left( Q + \frac{2}{k} \right) \frac{p - 1}{p}. \]

Now, we require \( \sigma \leq 0 \) which is equivalent to
\[ Q \leq 2 \left( 1 - \frac{1}{k} \right) + \frac{\gamma + 2}{p - 1}. \] (24)

In this case, the integral \( I(R) \), increasing in \( R \), is bounded uniformly w.r.t. \( R \).

The monotone convergence theorem implies that \( |\eta|_{H}^{\gamma} |u|^p \) belongs to \( L^1(\mathbb{R}^{2N+1}) \).

Note that instead of (23) we have more precisely
\[
\int |\eta|_{H}^{\gamma} |u|^p \varphi_R \, d\eta \, dt \leq ||a||_{L^\infty} \left( \int_{C_R} |\eta|_{H}^{\gamma} |u|^p \varphi_R \, d\eta \, dt \right)^{1/p} \left( \tilde{A}_{p,\gamma}(R)^{1/p'} + \tilde{B}_{p,\gamma}(R)^{1/p'} \right)
\]
\[
\leq C \int_{C_R} |\eta|_{H}^{\gamma} |u|^p \varphi_R \, d\eta \, dt,
\]
where \( C_R \) is defined in (9). Finally, using the dominated convergence theorem, we obtain that
\[
\lim_{R \to +\infty} \int_{C_R} |\eta|_{H}^{\gamma} |u|^p \varphi_R \, d\eta \, dt = 0.
\]

Hence
\[
\int |\eta|_{H}^{\gamma} |u|^p \, d\eta \, dt = 0,
\]
which implies that \( u \equiv 0 \). This contradicts the fact that \( u \) is a nontrivial weak solution of \( (I_k) \), which achieves the proof. \( \square \)

**Remark 3.** To determine the critical exponent for the hypoelliptic inequality
\[ -\Delta_H (au) \geq |\eta|_{H}^{\gamma} |u|^p, \]

it suffices to tend formally \( k \) to infinity and obtain \( 2 + \frac{\gamma + 2}{p - 1} \).

**Acknowledgments**

The authors are very grateful to Professor M. Kirane for interesting remarks and helpful discussion of the results.

**References**


