Existence and nonexistence results for higher-order semilinear evolution inequalities with critical potential

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Abstract

We prove nonexistence results for higher–order semilinear evolution equations and inequalities of the form $\frac{\partial^k u}{\partial t^k} - \Delta u + \frac{\lambda}{|x|^2} u \geq |u|^q$ in $\mathbb{R}^N \times (0, \infty)$, where $\lambda \geq -\left(\frac{N-2}{2}\right)^2$. This problem can be seen as a higher-order evolution version of the nonlinear Wheeler-De Witt equation which appears in the theory of quantum cosmology.

In order to show that our result is sharp in the parabolic case, we establish the existence of positive solutions to the semilinear equation $\frac{\partial u}{\partial t} - \Delta u + \frac{\lambda}{|x|^2} u \geq u^q$ in $\mathbb{R}^N \times (0, \infty)$, for $\lambda \geq 0$.

The nonexistence results are based on the test function method, developed by Mitidieri, Pohozaev, Tesei and Véron. The existence result is established by the construction of an explicit global solution of the semilinear parabolic inequality.

Key words: blow-up, nonexistence, differential inequality, potential

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1 Introduction

The problem of nonexistence of global solutions to nonlinear partial differential equations is of great interest. We cannot cite all the results in this direction and we refer the reader to [5,6,7,8,9,10,11,13] and the survey paper [1].

In this note we study the nonexistence of global solutions for higher-order (with respect to t) semilinear partial differential inequalities with critical potential

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u + \frac{\lambda}{|x|^2} u \geq |u|^q, & (x,t) \in \mathbb{R}^N \times (0,\infty), \quad \lambda \geq -\left(\frac{N-2}{2}\right)^2, \quad k \geq 1, \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0, & x \in \mathbb{R}^N, \quad N \geq 3. \end{cases} \tag{1}$$

The well-known operator $-\Delta + \frac{\lambda}{|x|^2}$ (see [22]) appears naturally in the nonlinear Wheeler-De Witt equation which deals with the minisuperspace model in quantum cosmology. We can refer the reader to [4] and the references therein for a complete description of the model. Our problem can be seen then as a higher-order evolution version of the nonlinear Wheeler-De Witt equation.

The nonexistence results hold for q less or equal than critical exponents which depend on N, λ and k. For the parabolic case ($k = 1$), we show that our result is sharp. For that, we construct a global nonnegative solution to the semilinear inequality

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\lambda}{|x|^2} u \geq |u|^q.$$

As for parabolic case ($k = 1$), the paper by Qi Zhang [21] contains an almost complete characterization (for positive solutions) of the relation between the critical exponent of the equation

$$\frac{\partial u}{\partial t} - \Delta u + V(x)u = u^q \tag{2}$$

and the potentials V behaving like $a/(1+|x|^b)$, where a and b are real numbers. The main Qi Zhang's result is that the relation is quantized depending on b and when $V(x) \sim a/(1+|x|^2)$ we are at a border-line case where the critical exponent can vary in $(1,\infty]$, and in all other cases the critical exponent takes only three values: 1, ∞ and $1+\frac{2}{N}$ (the classical Fujita number). So, $V(x)$ with $b = 2$ plays a critical role and is called "the critical potential".

Unfortunately, for general potential $V(x) \sim a/(1+|x|^2)$ it seems unrealistic to clear cut relation between the critical number and the potential. In the present paper we investigate very particular case [22] when

$$V(x) = \frac{\lambda}{|x|^2}.$$

We want to draw the reader's attention to the fact that we consider solutions without any sign assumption. The nonexistence results are based on the test function method, developed by Mitidieri, Pohozaev, Tesei and Véron [13,14,15,16,17].

Definition 1 *Let $u(x,t) \in C(\mathbb{R}^N \times [0,\infty))$ and the locally integrable traces $\frac{\partial^i u}{\partial t^i}(x,0)$, $i = 1,\dots,k-1$, are well defined. The function $u(x,t)$ is called a weak solution to problem (1) if, for any nonnegative test-function $\varphi(x,t)$, $\frac{\partial^k \varphi}{\partial t^k} \in C(\mathbb{R}^N \times [0,\infty))$, $-\Delta\varphi + \frac{\lambda}{|x|^2}\varphi \in L_1(\mathbb{R}^N \times (0,\infty))$ with compact support, the integral inequality*

$$\int_0^\infty \int_{\mathbb{R}^N} u \left((-1)^k \frac{\partial^k \varphi}{\partial t^k} - \Delta\varphi + \frac{\lambda}{|x|^2}\varphi \right) dxdt \geq \int_0^\infty \int_{\mathbb{R}^N} |u|^q \varphi \, dxdt$$
$$+ \sum_{i=1}^{k-1} (-1)^i \int_{\mathbb{R}^N} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x,0) \frac{\partial^i \varphi}{\partial t^i}(x,0)\, dx + \int_{\mathbb{R}^N} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0)\varphi(x,0)\, dx \quad (3)$$

holds.

Let us introduce the parameters

$$s^* = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda},\ s_* = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda}. \quad (4)$$

Note that s^* and s_* verify

$$\left(-\Delta + \frac{\lambda}{|x|^2}\right)|x|^{s_*} = \left(-\Delta + \frac{\lambda}{|x|^2}\right)|x|^{-s^*} = 0, \text{ for any } x \in \mathbb{R}^N \setminus \{0\}.$$

Our result for problem (1) is the following

Theorem 2 *Let*

$$\lambda \geq 0 \quad \text{and} \quad 1 < q \leq q^* = 1 + \frac{2}{s^* + 2/k}$$

or

$$0 > \lambda \geq -\left(\frac{N-2}{2}\right)^2 \quad \text{and} \quad 1 < q \leq q^* = 1 + \frac{2}{-s_* + 2/k}.$$

Then the problem (1) has no nontrivial global solution.

2 Auxiliary estimates

In this section we obtain some estimates depending on the parameter ρ, $\rho \to \infty$. These estimates play a fundamental role in the test function method [13,14,15,16,17].

Let us consider the "standard cut-off function" $\zeta(y) \in C^\infty(\mathbb{R}_+)$ with the following properties:

$$0 \le \zeta(y) \le 1, \qquad \zeta(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1, \\ 0 & \text{if } y \ge 2. \end{cases}$$

For the function

$$\eta(y) = (\zeta(y))^{kp_0}$$

with some positive $p_0 > 1$ and $k \in \mathbb{N}$, by direct calculation one can obtain the estimates (for $1 < p \le p_0$)

$$\begin{aligned} &|\eta'(y)|^p = (kp_0)^p \zeta^{kp_0(p-1)} \zeta^{kp_0-p} |\zeta'|^p \le c_\eta \eta^{p-1}(y), \\ &|\eta''(y)|^p \le (kp_0)^p \zeta^{kp_0(p-1)} \zeta^{kp_0-2p}((kp_0-1)|\zeta'|^2 + \zeta|\zeta''|)^p \le c_\eta \eta^{p-1}(y), \quad \ldots \\ &|\eta^{(k)}(y)|^p \le c_\eta \eta^{p-1}(y), \end{aligned}$$

with a positive constant c_η.
Now let us introduce the change of variables $y = t/\rho^\theta$, with $\theta > 0$, $\rho > 1$. For the function $\eta(t/\rho^\theta)$ we have

$$\operatorname{supp} \left|\eta\left(\frac{t}{\rho^\theta}\right)\right| = \{0 \le t \le 2\rho^\theta\}, \;\; \operatorname{supp} \left|\frac{d^k \eta(t/\rho^\theta)}{dt^k}\right| = \{\rho^\theta \le t \le 2\rho^\theta\},$$

and

$$\int_{\operatorname{supp}\left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right|} \frac{\left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right|^p}{\eta^{p-1}(t/\rho^\theta)}\, dt \le c_\eta \rho^{-\theta(kp-1)}. \tag{5}$$

The parameter θ will be chosen later.
For the variable x we introduce the functions $\eta(r/\rho)$,

$$\xi(x) \equiv \xi(r) = r^s, \tag{6}$$

and

$$\psi_\rho(x) \equiv \psi_\rho(r) = r^s \eta\left(\frac{r}{\rho}\right). \tag{7}$$

For the derivatives of the function $\psi_\rho(r)$ we have:

$$\begin{aligned} \left|\frac{\partial \psi_\rho}{\partial r}\right|^p &\le \left| s r^{s-1} \eta\left(\frac{r}{\rho}\right) + r^s \eta'\left(\frac{r}{\rho}\right)\frac{1}{\rho}\right|^p \\ &\le c\eta^{p-1}\left(\frac{r}{\rho}\right) r^{(s-1)p}\left(1 + \frac{r^p}{\rho^p}\right), \\ \left|\frac{\partial^2 \psi_\rho}{\partial r^2}\right|^p &\le \left| s(s-1) r^{s-2} \eta\left(\frac{r}{\rho}\right) + \frac{2sr^{s-1}}{\rho}\eta'\left(\frac{r}{\rho}\right) + \frac{r^s}{\rho^2}\eta''\left(\frac{r}{\rho}\right)\right|^p \\ &\le c\eta^{p-1}\left(\frac{r}{\rho}\right) r^{(s-2)p}\left(1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}}\right), \end{aligned}$$

here c does not depend on r and ρ. Using these estimates we arrive at the inequality for the operator $A = \Delta - \frac{\lambda}{|x|^2}$:

$$\begin{aligned}
|A\psi_\rho(x)|^p &= \left|\Delta\psi_\rho(x) - \frac{\lambda}{|x|^2}\psi_\rho(x)\right|^p \\
&= \left|\frac{\partial^2\psi_\rho}{\partial r^2} + \frac{N-1}{r}\frac{\partial\psi_\rho}{\partial r} - \frac{\lambda}{r^2}\psi_\rho(x)\right|^p \\
&\le c\left|\frac{\partial^2\psi_\rho}{\partial r^2}\right|^p + c\frac{1}{r^p}\left|\frac{\partial\psi_\rho}{\partial r}\right|^p + c\frac{1}{r^{2p}}|\psi_\rho|^p \\
&\le c\eta^{p-1}\left(\frac{r}{\rho}\right) r^{(s-2)p}\left(1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}}\right) \\
&\le c\psi_\rho^{p-1}(x) r^{s-2p}\left(1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}}\right).
\end{aligned} \tag{8}$$

If $\lambda \ge 0$ then we choose $s = s_* \ge 0$. Due to $A(r^{s_*}) = 0$, we have $A\psi_\rho = 0$ for $r \le \rho$ and $\operatorname{supp}|A\psi_\rho| \subset \{\rho \le r \le 2\rho\}$. On the set $\operatorname{supp}|A\psi_\rho|$ the estimate $1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}} \le c$ holds, where c does not depend on r and ρ. Therefore, it follows from (8) (for $\rho \le r \le 2\rho$) that

$$|A\psi_\rho(x)|^p \le c\psi_\rho^{p-1}(x)\rho^{s_*-2p};$$

and we get

$$\int_{\operatorname{supp}|A\psi_\rho|} \frac{|A\psi_\rho(x)|^p}{\psi_\rho^{p-1}(x)}\,dx \le c\int_\rho^{2\rho} \frac{\psi_\rho^{p-1}(x)}{\psi_\rho^{p-1}(x)}\rho^{s_*-2p} r^{N-1}\,dr \le c_\psi \rho^{s_*-2p+N}. \tag{9}$$

For the general test-function

$$\varphi_\rho(x,t) = \eta\left(\frac{t}{\rho^\theta}\right)\psi_\rho(x) \tag{10}$$

we obtain the inequality

$$\begin{aligned}
\int_{\operatorname{supp}|A\varphi_\rho|} \frac{|A\varphi_\rho(x,t)|^p}{\varphi_\rho^{p-1}(x,t)}\,dx\,dt &\le \int_0^{2\rho^\theta} \eta(t/\rho^\theta)\,dt \int_{\operatorname{supp}|A\psi_\rho|} \frac{|A\psi_\rho|^p}{\psi_\rho^{p-1}}\,dx \\
&\le c_\varphi \rho^{\theta+s_*-2p+N}.
\end{aligned} \tag{11}$$

Analogously, using (5), we obtain:

$$
\begin{aligned}
&\int_{\operatorname{supp}\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|}\left|\frac{\partial^k\varphi_\rho(x,t)}{\partial t^k}\right|^p\frac{1}{\varphi_\rho^{p-1}(x,t)}\,dx\,dt\\
&\le\int_{\operatorname{supp}\left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right|}\left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right|^p\frac{1}{\eta^{p-1}(t/\rho^\theta)}\,dt\int_{0<|x|<2\rho}\psi_\rho(x)\,dx\\
&\le c_\eta\rho^{-\theta(kp-1)}c\int_0^{2\rho}r^{s_*+N-1}\,dr\le c_\varphi\rho^{s_*+N-\theta(kp-1)}. \qquad (12)
\end{aligned}
$$

For $\theta=2/k$, the powers in these two estimates are equal:

$$
s_*+N-\theta(kp-1)=\theta+s_*-2p+N\equiv s_*-2p+N+2/k.
$$

Finally, we have

$$
J_p\equiv\int_{\operatorname{supp}|(-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k}-A\varphi_\rho|}\frac{\left|(-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k}-A\varphi_\rho\right|^p}{\varphi_\rho^{p-1}}\,dxdt\le c_0\rho^{s_*-2p+N+2/k}. \qquad (13)
$$

If $-\left(\frac{N-2}{2}\right)^2\le\lambda<0$, we take $s=-s^*\le 0$. Therefore, $A(r^{-s^*})=0$, $A\psi_\rho=0$ for $r\le\rho$ and $\operatorname{supp}|A\psi_\rho|=\{\rho\le r\le 2\rho\}$,

$$
\int_{\operatorname{supp}|A\psi_\rho|}\frac{|A\psi_\rho(x)|^p}{\psi_\rho^{p-1}(x)}\,dx\le c\int_\rho^{2\rho}\frac{\psi_\rho^{p-1}(x)}{\psi_\rho^{p-1}(x)}\rho^{-s^*-2p}r^{N-1}\,dr\le c_\psi\rho^{-s^*-2p+N} \qquad (14)
$$

and

$$
J_p\equiv\int_{\operatorname{supp}|(-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k}-A\varphi_\rho|}\frac{\left|(-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k}-A\varphi_\rho\right|^p}{\varphi_\rho^{p-1}}\,dxdt\le c_0\rho^{-s^*-2p+N+2/k}. \qquad (15)
$$

3 Proof of theorem 2

Let $u(x,t)$ be a global nontrivial solution of problem (1). From Definition 1 with the test function $\varphi(x,t)=\varphi_\rho(x,t)$, defined by (10) with $p=q'>1$, $s=s_*$ or $s=-s^*$, and $\theta=2/k$, using the equalities $\frac{\partial^i\varphi_\rho}{\partial t^i}(x,0)\equiv 0$, $i=1,\dots,k-1$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx+\int_0^\infty\int_{\mathbb{R}^N}|u|^q\varphi_\rho\,dxdt\\
&\le\int_{\operatorname{supp}\left|(-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k}-A\varphi_\rho\right|}u\left((-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k}-A\varphi_\rho\right)\,dxdt. \qquad (16)
\end{aligned}
$$

As for the last integral in (16), using the Hölder inequality, we find that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx + \int_0^\infty \int_{\mathbb{R}^N} |u|^q \varphi_\rho \,dxdt \\
&= \int_{\mathbb{R}^N} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx + \int_{\operatorname{supp}\left|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho\right|} |u|^q \varphi_\rho \,dxdt \\
&+ \int_{\varphi_\rho(x,t)=\xi(x)} |u|^q \xi \,dxdt \\
&\leq \int_{\operatorname{supp}|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho|} |u| \cdot \left|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho\right| dxdt \\
&\leq \left(\int_{\operatorname{supp}\left|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho\right|} |u|^q \varphi_\rho \,dxdt \right)^{1/q} J_{q'}^{1/q'}. \qquad (17)
\end{aligned}
$$

If $\lambda \geq 0$, using the estimate (13) (with $p = q'$), we have

$$
\int_{\varphi_\rho(x,t)=\xi(x)} |u|^q \xi(x)\,dxdt \leq J_{q'} \leq c_0 \rho^{s_* - 2q' + N + 2/k}. \qquad (18)
$$

Now we pass to the limit as $\rho \to \infty$. In the case

$$
s_* - 2q' + N + 2/k \leq 0 \qquad (19)
$$

this implies

$$
\int_0^\infty \int_{\mathbb{R}^N} |u|^q \xi \,dxdt \leq c_0.
$$

Then by the inequality $\varphi_\rho \leq \xi$ and taking into account the general properties of Lebesgue integral we have

$$
\int_{\operatorname{supp}\left|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho\right|} |u|^q \varphi_\rho \,dxdt \leq \int_{\operatorname{supp}\left|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho\right|} |u|^q \xi \,dxdt = \varepsilon(\rho) \to 0
$$

as $\rho \to \infty$. Then from the inequality (17) we finally get

$$
\int_{\varphi_\rho(x,t)=\xi(x)} |u|^q \xi \,dxdt \leq \varepsilon^{1/q}(\rho) c_0^{1/q'} \to 0
$$

as $\rho \to \infty$, and $\int_0^\infty \int_{\mathbb{R}^N} |u|^q \xi \,dxdt = 0$, that is, the solution $u(x,t)$ must be trivial under condition (19), which is equivalent to the condition of Theorem 2 for $\lambda \geq 0$.
For $\lambda < 0$ we take $s = -s^*$ and the estimate (15) (with $p = q'$) gives

$$
\int_{\varphi_\rho(x,t)=\xi(x)} |u|^q \xi(x)\,dxdt \leq J_{q'} \leq c_0 \rho^{-s^* - 2q' + N + 2/k}.
$$

In the case $-s^* - 2q' + N + 2/k \leq 0$, it implies the nonexistence of nontrivial solution.

4 Generalizations

In this section we present possible generalizations. For simplicity we restrict our attention to the case $\lambda \geq 0$.
Let us consider the inhomogeneous problem [20]

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u + \frac{\lambda}{|x|^2} u \geq |u|^q + w(x), & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{20}$$

with $w(x) \in L^1_{\text{loc}}(\mathbb{R}^N)$, $w(x) \geq 0$. We understand the weak solution of this problem in the sense of Definition 1 with the extra term $\int_0^\infty \int_{\mathbb{R}^N} w(x)\varphi \, dxdt$.

Theorem 3 *Let $\lambda \geq 0$ and*

$$1 < q < \frac{s^* + 2}{s^*} = 1 + \frac{2}{s^*}.$$

Then the problem (20) has no nontrivial global solution for any arbitrary small $w(x) \geq 0$, $w(x) \not\equiv 0$.

PROOF. Following the proof of Theorem 2, we obtain the a priori estimate

$$\int_0^{\rho^{2/k}} \int_{\{|x|<\rho\}} w(x)\xi \, dx \, dt + \int_{\{|x|<\rho\}} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0)\xi(x) \, dx$$
$$+ \iint_{\varphi_\rho(x,t)=\xi(x)} |u|^q \xi \, dx \, dt \ \leq \ c_1 \rho^{-2q' + s_* + N + 2/k};$$

hence

$$c_1 \rho^{-2q' + s_* + N + 2/k} \geq \int_0^{\rho^{2/k}} \int_{\{|x|<\rho\}} w(x)\xi(x) \, dx \, dt \geq \rho^{2/k} c_w \tag{21}$$

with ρ such, that

$$\int_{\{|x|<\rho\}} w(x)\xi(x) \, dx \geq c_w \equiv \text{const} > 0.$$

If $-2q' + s_* + N < 0$, then the inequality (21) cannot hold as $\rho \to \infty$.

Finally, we extend our nonexistence results to the following systems

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u + \frac{\lambda}{|x|^2} u \geq |v|^{q_1}, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ \frac{\partial^k v}{\partial t^k} - \Delta v + \frac{\lambda}{|x|^2} v \geq |u|^{q_2}, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0, \quad \frac{\partial^{k-1} v}{\partial t^{k-1}}(x,0) \geq 0, \quad x \in \mathbb{R}^N. \end{cases} \tag{22}$$

We precise that the sharpness of the result which follows is not established.

Theorem 4 *Let $\lambda \geq 0$, $q_1 > 1$, $q_2 > 1$ and*

$$\max\{\gamma_1, \gamma_2\} \geq \frac{s^* + 2/k}{2}, \quad \textit{where} \quad \gamma_1 = \frac{q_1+1}{q_1 q_2 - 1}, \quad \gamma_2 = \frac{q_2+1}{q_1 q_2 - 1}.$$

Then (22) has no nontrivial global solution.

PROOF. In this theorem the concept of solution is understood in the weak sense of Definition 1. Using the Hölder inequality, from the definition we obtain

$$\int_{\mathbb{R}^N} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \varphi_\rho(x,0)\, dx + \int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1} \varphi_\rho \, dxdt \leq \left(\iint_{\text{supp}\left|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho\right|} |u|^{q_2} \varphi_\rho \, dxdt \right)^{1/q_2} J_{q_2}^{1/q_2'}, \tag{23}$$

$$\int_{\mathbb{R}^N} \frac{\partial^{k-1} v}{\partial t^{k-1}}(x,0) \varphi_\rho(x,0)\, dx + \int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2} \varphi_\rho \, dxdt \leq \left(\iint_{\text{supp}\left|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho\right|} |v|^{q_1} \varphi_\rho \, dxdt \right)^{1/q_1} J_{q_1}^{1/q_1'}, \tag{24}$$

where from (13) we have

$$J_{q_1} \leq c_0 \rho^{-2q_1' + s_* + N + 2/k}, \qquad J_{q_2} \leq c_0 \rho^{-2q_2' + s_* + N + 2/k}.$$

We substitute (24) in (23). Then

$$\int_{\mathbb{R}^N} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \psi_\rho(x)\, dx + \int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1} \varphi_\rho \, dxdt \leq \left(\iint_{\text{supp}\left|(-1)^k \frac{\partial^k \varphi_\rho}{\partial t^k} - A\varphi_\rho\right|} |v|^{q_1} \varphi_\rho \, dxdt \right)^{1/(q_1 q_2)} J_{q_1}^{1/(q_1' q_2)} J_{q_2}^{1/q_2'},$$

hence

$$\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1}\varphi_\rho \, dxdt \le \left(J_{q_1}^{q_1-1} J_{q_2}^{q_1(q_2-1)}\right)^{1/(q_1q_2-1)} \le C\rho^{s^*+2/k-2\gamma_1}. \tag{25}$$

Therefore, there is no nontrivial $v(x,t)$ if

$$s^* + 2/k - 2\gamma_1 \le 0.$$

Analogously, substituting (23) to (24), we deduce

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2}\varphi_\rho \, dxdt \le \left(J_{q_2}^{q_2-1} J_{q_1}^{q_2(q_1-1)}\right)^{1/(q_1q_2-1)} \le C\rho^{s^*+2/k-2\gamma_2}, \tag{26}$$

so that there is no nontrivial $u(x,t)$ if $s^* + 2/k - 2\gamma_2 \le 0$. It is evidently that if $u(x,t) \equiv 0$, then $v(x,t) \equiv 0$, and vice versa. Finally we get the general nonexistence condition

$$\max\{\gamma_1, \gamma_2\} \ge \frac{s^* + 2/k}{2}.$$

5 Existence Results

In this section, we will limit ourselves to the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \frac{\lambda}{|x|^2} u \ge |u|^q & \text{in } \mathbb{R}^N \times]0, +\infty[, \\ \\ u(x,0) = u_0(x) \ge 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{27}$$

with $\lambda \ge 0$. In this case, Theorem 1 shows that if

$$1 < q \le q^* = 1 + \frac{2}{N + s_*}$$

then the problem (27) has no global nontrivial solution [3,12]. We complete this nonexistence result by the existence one, [19,2,12]:

Theorem 5 *If*

$$q > q^* \equiv 1 + \frac{2}{N + s_*}$$

then nontrivial global solutions of (27) *exist.*

PROOF. Let v be a positive solution of

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v + \frac{\lambda}{|x|^2} v = 0 \quad \text{in} \quad \mathbb{R}^N \times]0, +\infty[, \\ \\ v(x,0) = v_0(x) \geq 0 \quad \text{in} \quad \mathbb{R}^N, \end{cases}$$

and let the function w defined on $\mathbb{R}^N \times]0, +\infty[$ by $w(x,t) = \alpha(t) v(x,t)$, where the function α has to be defined. If α is selected such that

$$\alpha'(t) = (\alpha(t))^q \left[\sup_{x \in \mathbb{R}^N} v(x,t) \right]^{q-1} = (\alpha(t))^q ||v(.,t)||^{q-1}_{L^\infty(\mathbb{R}^N)},$$

then w is a solution of (27) on its interval of definition. Let us set α be the solution of the Cauchy problem

$$\begin{cases} \alpha'(t) = (\alpha(t))^q ||v(.,t)||^{q-1}_{L^\infty(\mathbb{R}^N)}, \quad t > 0, \\ \\ \alpha(0) = \alpha_0 > 0. \end{cases} \tag{28}$$

It is easy to see that the solution of (28) is global if, and only if,

$$\int_0^{+\infty} ||v(.,t)||^{q-1}_{L^\infty(\mathbb{R}^N)} \, dt < +\infty \tag{29}$$

and

$$0 < \alpha_0 < \left((q-1) \int_0^{+\infty} ||v(.,t)||^{q-1}_{L^\infty(\mathbb{R}^N)} \, dt \right)^{\frac{-1}{q-1}}.$$

At this stage, we will construct the function v on $\mathbb{R}^N \times]0, +\infty[$. Consider the function v defined on $\mathbb{R}^N \times]0, +\infty[$ by

$$v(x,t) = \frac{1}{t+1} \frac{1}{|x|^{\frac{1}{2}(N-2)}} I_\nu \left(\frac{|x|}{2(t+1)} \right) \exp\left(-\frac{|x|^2 + 1}{4(t+1)} \right),$$

where $\nu = s_* + \frac{N-2}{2}$ and I_ν is the modified Bessel function of order ν [18]. One can observe by direct calculation that the function v is a positive solution of

$$\frac{\partial v}{\partial t} - \Delta v + \frac{\lambda}{|x|^2} v = 0 \;\; \text{in} \;\; \mathbb{R}^N \times]0, +\infty[.$$

Recall that the asymptotic behaviour of I_ν in the neighborhood of 0 and $+\infty$ is given respectively by [18]

$$I_\nu(z) \sim \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \quad \text{as} \quad z \longrightarrow 0^+ \tag{30}$$

and

$$I_\nu(z) \sim \frac{\exp(z)}{\sqrt{2\pi z}} \quad \text{as} \quad z \longrightarrow +\infty. \tag{31}$$

Morover, if the following estimate

$$\limsup_{t\to+\infty} \, (t+1)^{\frac{1}{q^*-1}} ||v(.,t)||_{L^\infty(\mathbb{R}^N)} < +\infty \tag{32}$$

holds then the condition (29) will be satisfied for any $q > q^*$. Indeed, it suffices to remark that

$$\int_0^{+\infty} (t+1)^{-\frac{q-1}{q^*-1}} \, dt < +\infty, \quad \text{for any} \quad q > q^*.$$

We will show now that the esimate (32) is satisfied. First, since

$$\lim_{|x|\to 0} v(x,t) = \lim_{|x|\to\infty} v(x,t) = 0, \quad \forall \, t > 0,$$

then, for any $t > 0$, there exists $r^*(t) \in]0, +\infty[$, such that

$$v(r^*(t), t) = ||v(.,t)||_{L^\infty(\mathbb{R}^N)}.$$

Let

$$\mathcal{V}(t) = (t+1)^{\frac{1}{q^*-1}} \; v(r^*(t), t).$$

Using the fact that

$$\frac{1}{q^*-1} = \frac{N+s_*}{2},$$

we can write

$$\mathcal{V}(t) = (t+1)^{\frac{s_*}{2}} \left(\frac{r^*(t)}{t+1}\right)^{-\frac{N-2}{2}} I_\nu\left(\frac{r^*(t)}{2(t+1)}\right) \exp\left(-\frac{(r^*(t))^2+1}{4(t+1)}\right).$$

If we set

$$y^*(t) = \frac{r^*(t)}{2(t+1)},$$

then

$$\mathcal{V}(t) = e^{-\frac{1}{4(t+1)}} (t+1)^{\frac{s_*}{2}} (y^*(t))^{-\frac{N-2}{2}} I_\nu(y^*(t)) e^{-(t+1)(y^*(t))^2}.$$

Suppose that there is a sequence $(t_k)_{k\in\mathbb{N}} \to +\infty$ such that

$$\lim_{t_k\to+\infty} \mathcal{V}(t_k) = +\infty.$$

Three cases can arise:

- Case 1: There exists a subsequence, also denoted by $(t_k)_{k\in\mathbb{N}}$, such that

$$\lim_{t_k\to+\infty} \; y^*(t_k) = +\infty.$$

In this case

$$\mathcal{V}(t_k) \sim \text{const} \cdot (t_k+1)^{\frac{s_*}{2}} (y^*(t_k))^{-\frac{N-1}{2}} e^{y^*(t_k)} e^{-(t_k+1)(y^*(t_k))^2} \quad \text{as} \quad t_k \to +\infty,$$

which implies that

$$\lim_{t_k \to \infty} \mathcal{V}(t_k) = 0.$$

• Case 2: There exists a subsequence, also denoted by $(t_k)_{k\in\mathbb{N}}$, such that

$$\lim_{t_k \to +\infty} y^*(t_k) = 0.$$

In this case

$$\mathcal{V}(t_k) \sim \text{const} \cdot \left[(t_k+1)(y^*(t_k))^2\right]^{\frac{s_*}{2}} e^{-(t_k+1)(y^*(t_k))^2} \quad \text{as} \quad t_k \to +\infty,$$

which implies that $\mathcal{V}(t_k)$ is bounded, since the function $z \longmapsto z^{\frac{s_*}{2}} e^{-z}$ is bounded on $\mathbb{R}^+$.

• Case 3: There are two constatnts A and B such that the sequence $(y^*(t_k))_{k\in\mathbb{N}}$ satisfies

$$0 < A \leq y^*(t_k) \leq B < +\infty.$$

In this case, the expression $\mathcal{V}(t_k)$ is clearly bounded.
Whence, there is no subsequence of $(t_k)_{k\in\mathbb{N}} \to +\infty$ such that

$$\lim_{t_k \to +\infty} \mathcal{V}(t_k) = +\infty,$$

which implies that there is no sequence $(t_k)_{k\in\mathbb{N}} \to +\infty$ such that

$$\lim_{t_k \to +\infty} \mathcal{V}(t_k) = +\infty.$$

This ends the proof.

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