Existence of solution for a anisotropic equation with critical exponent

Claudianor O. Alves* and Abdallah El Hamidi †

Abstract

Using variational methods we establish existence of nontrivial solutions for the following class of anisotropic critical problem

$$(P_\lambda) \qquad \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = \lambda f(u) + g(u), \quad \text{in} \quad \Omega \\ u \geq 0, \quad \text{in} \quad \Omega \\ u = 0, \quad \text{on} \quad \partial\Omega \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, λ is a positive parameter, $g(u)$ behaves like $|u|^{p^*-2}u$, p^* is the critical exponent for this class of problem and f is a continuous function verifying some adequate assumptions.

Keywords. Anisotropic Sobolev spaces, critical exponent, Palais-Smale

Mathematics subject Classification. 35J70; 46E35 ; 35B65

1 Introduction

In this paper, we are interested in the existence of solutions to a class of anisotropic elliptic equations involving critical exponents. More precisely, we

*The author is supported by CNPq - Brazil, email: coalves@dme.ufcg.edu.br

†email: aelhamid@univ-lr.fr

study the existence of nontrivial solutions for the following class of anisotropic critical problems

$$(P_\lambda) \qquad \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = \lambda f(u) + g(u), & \text{in} \quad \Omega \\ u \geq 0, \quad \text{in} \quad \Omega \\ u = 0, \quad \text{on} \quad \partial\Omega \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, λ is a positive parameter, the exponents p_i satisfy

$$p_i > 1, \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1.$$

The functions f and g are assumed to be continuous and satisfy some growth conditions which will be stated below.

This work is motivated by the recent results by Fragala *et al.* [4] and by the first author and Rakotoson [3, 2]. Indeed, In [4], the authors have considered the same anisotropic differential operator as in (P_λ) and established existence and regularity results in the *subcritical* case. Also, nonexistence results, based on a modification of the *starsharpness* of domains, are showed.

In [3], the authors have given a generalization of the well-known concentration-compactness principle of Lions [5] in the anisotropic case. Using this new concentration-compactness principle, they showed that a certain critical best Sobolev constant is achieved. In [2], the authors considered a perturbed critical anisotropic problem and showed some existence and regularity results in the whole space.

In the present work, we consider a more general class of problems (P_λ) in the case of bounded domains in $\mathbb{R}^N$. Two types of growth conditions on the function f are considered here, the sublinear and the superlinear growth conditions.

Let us notice here that only few results dealing with critical anisotropic problems can be found in the literature.

The natural functional framework of Problem $(\mathcal{P}_\lambda)$ is the anisotropic Sobolev spaces theory developed by [6, 12, 7, 8, 11]. Then, let $W_0^{1,\vec{p}}(\Omega)$ be the completion of the space $\mathcal{D}(\Omega)$ with respect to the norm $\|u\|_{1,\vec{p}} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}$.
It is well known that $\left(W_0^{1,\vec{p}}(\Omega), \|\cdot\|_{1,\vec{p}}\right)$ is a reflexive Banach space which is continuously embedded in $L^{p^*}(\Omega)$, where p^* is defined by

$$p^* = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i} - 1}.$$

In what follows, we will assume that

$$\max\{p_1,\ p_2,\ ...,\ p_N\} < p^*.$$

Under this condition, the exponent p^* is the effective critical exponent for the anisotropic differential operator defined in $(\mathcal{P}_\lambda)$, we refer the reader to [4] for more details about the anisotropic critical exponent.
The following notations will be adopted in the sequel: $p_- = \min\{p_1,\ p_2,\ ...,\ p_N\}$, $p_+ = \max\{p_1,\ p_2,\ ...,\ p_N\}$ and $\vec{p} = (p_1,\ p_2,\ \cdots,\ p_N)$. Also, $\|\cdot\|_{p_i}$ will denote the usual Lebesgue norm in $L^{p_i}(\mathbb{R}^N)$.

In the first result, we assume that f and g satisfy the following hypotheses:

(f_1) There exist positive constants $C_1, C_2 > 0$ and $q \in (0, p_-)$ such that

$$C_1 t^{q-1} \le f(t) \le C_2 t^{q-1}, \quad \forall\, t \ge 0.$$

(g_1) There exists $C_3 > 0$ such that $|g(t)| \le C_3 |t|^{p^*-1}, \quad \forall\, t \in \mathbb{R}$.

(g_2) There exists $\gamma \in (p_+, p^*]$ such that

$$0 < \gamma G(t) \le t\, g(t), \quad \forall\, t > 0.$$

where $G(t) = \int_0^t g(s)\,ds$.

Theorem 1. *If* (f_1), (g_1) *and* (g_2) *hold true, then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$*, problem* (P_λ) *has a nontrivial solution.*

It is clear that for every $t \geq 0$, we get

$$-\frac{C_3}{p^*} t^{p^*} \leq G(t) \leq \frac{C_3}{p^*} t^{p^*}.$$

In the second result, we are concerned with the special case $g(u) = |u|^{p^*-2}u$, we assume the following conditions on f:

(f_2) $\displaystyle \lim_{t \longrightarrow 0^+} \frac{f(t)}{t^{p_+-1}} = 0$

(f_3) There exists $q \in (p_+, p^*)$ such that

$$\limsup_{t \longrightarrow +\infty} \frac{f(t)}{t^{q-1}} = l < +\infty.$$

(f_4) For all $v \in W_0^{1,\overrightarrow{p}}(\Omega) \setminus \{0\}$ the function

a)
$$t \mapsto \frac{\lambda \int_\Omega f(tv)tv + t^{p^*} \|v\|_{p^*}^{p^*}}{\sum_{i=1}^{N} a_i t^{p_i}}$$

is increasing in the interval $(0, +\infty)$, where

$$a_i = \left\| \frac{\partial v}{\partial x_i} \right\|_{p_i}^{p_i}$$

and

b)
$$\frac{\int_\Omega f(tv)tv}{\sum_{i=1}^{N} a_i t^{p_i}} \longrightarrow 0 \Longleftrightarrow t \longrightarrow 0.$$

(f_5) There exist a positive constant $\theta \in (p_+, q)$ and $t_0 > 0$ such that

$$0 < \theta F(t) \leq f(t)t, \quad \forall t \geq t_0$$

where

$$F(t) = \int_0^t f(s)\, ds. \tag{1.1}$$

Remark 1. The function $f(t) = |u|^{q_1-2}u + |u|^{q_2-2}u$ for $q_1, q_2 \in (p_+, p^*)$, verifies the hypotheses $(f_2) - (f_5)$.

Theorem 2. *If $(f_2) - (f_5)$ hold true, then there exists $\lambda^* > 0$ such that (P_λ) has at least a nontrivial solution for $\lambda \geq \lambda^*$.*

2 Proof of Theorem 1.

The proof of Theorem 1 will be divided into several lemmas. Hereafter, we will work with the space $W_0^{1,\overrightarrow{p}}(\Omega)$ where $\overrightarrow{p} = (p_1, ..., p_N)$ endowed with the following norm

$$\|u\| = \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}$$

where $\| \ \|_t$ denotes the norm of the $L^t(\Omega)$ space.
Related to (P_λ), we have the functional $I_\lambda : W_0^{1,\overrightarrow{p}}(\Omega) \longrightarrow \mathbb{R}$ given by

$$I_\lambda(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \lambda \int_\Omega F(u_+) dx - \int_\Omega G(u_+) dx,$$

where $u_+(x) = \max\{u(x), 0\}$ and F is defined by (1.1).
Once that, we have the following continuous embedding

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

using the fact that Ω is bounded, we get the continuous embedding

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^s(\Omega), \quad \forall s \in [1, p^*],$$

which implies that $I_\lambda \in C^1(W_0^{1,\overrightarrow{p}}(\Omega), \mathbb{R})$.

Lemma 1. *There exists $\lambda^* > 0$ and $r, \rho > 0$ such that for all $\lambda \geq \lambda^*$ we have*

$$I_\lambda(u) \geq r \quad \textit{for} \quad \|u\| = \rho.$$

Proof. For $\rho > 0$ sufficiently small, we get

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{p_i} \leq 1 \quad \forall i \in \{1, ..., N\}.$$

Thus,

$$I_\lambda(u) \geq \frac{1}{p_+} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_+} - \frac{C_2 \lambda}{q} \int_\Omega (u_+)^q dx - \frac{C_3}{p^*} \int_\Omega (u_+)^{p^*} dx,$$

and by above Sobolev embedding, there exist positive constants, $k_1, k_2, k_3 > 0$ such that

$$I_\lambda(u) \geq k_1 \|u\|^{p_+} - k_2 \lambda \|u\|^q - k_3 \|u\|^{p^*},$$

and this last inequality gives the result. ■

Lemma 2. *The functional I_λ is bounded from below in*

$$\bar{B}_\rho(0) = \left\{ u \in W_0^{1,\overrightarrow{p}}(\Omega);\ \|u\| \leq \rho \right\}.$$

Moreover,

$$I_\infty = \inf_{u \in \bar{B}_\rho(0)} I_\lambda(u) < 0, \quad \forall \lambda \in (0, \lambda^*). \tag{2.2}$$

Proof. Using the definition of I_λ, it is easy to check that I_λ is bounded from below in $\bar{B}_\rho(0)$. To prove (2.2), fix a nonnegative function $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \setminus \{0\}$ and note that for $t > 0$, we have

$$I_\lambda(t\varphi) \leq \sum_{i=1}^{N} \int_\Omega \frac{t^{p_i}}{p_i} \left| \frac{\partial \varphi}{\partial x_i} \right|^{p_i} dx - \frac{C_1 \lambda t^q}{q} \int_\Omega \varphi^q dx + \frac{C_3 t^{p^*}}{p^*} \int_\Omega \varphi^{p^*} dx.$$

Once that $q < p_- < p^*$, the last inequality implies that for some $t_1 > 0$ sufficiently small

$$I_\lambda(t_1\varphi) < 0 \quad \text{and} \quad t_1\varphi \in \overline{B}_\rho(0)$$

from where follows the lemma. ∎

Now, applying the Ekeland's Variational Principle to the functional I_λ on the metric space $\left(\bar{B}_\rho(0), d\right)$ endowed with the metric d given by

$$d(u,v) = \|u - v\|$$

there exists $(u_n)_n \subset \overline{B}_\rho(0)$ such that

$$I_\lambda(u_n) \longrightarrow I_\infty = \inf_{u \in \overline{B}_\rho} I_\lambda(u) \tag{2.3}$$

and

$$I_\lambda(v) - I_\lambda(u_n) \geq -\frac{1}{n}\|v - u_n\| \quad \forall\, v \neq u_n. \tag{2.4}$$

Since I_λ is differentiable, it follows from (2.4) that

$$I'_\lambda(u_n) \longrightarrow 0. \tag{2.5}$$

From (2.4) and (2.5), we get

$$I_\lambda(u_n) \longrightarrow I_\infty \quad \text{and} \quad I'_\lambda(u_n) \longrightarrow 0 \tag{2.6}$$

from where we can conclude that $(u_n)_n$ is a bounded $(PS)_{I_\infty}$ sequence to I_λ. Passing, if necessary, to a subsequence still denoted by $(u_n)_n$, we may assume that $(u_n)_n$ has a weak limit $u_\lambda \in W_0^{1,\overrightarrow{p}}(\Omega)$. Moreover, from the definition of the functional I_λ, we can assume that $(u_n)_n$ is a sequence of nonnegative functions.

Lemma 3. *The weak limit u_λ of $(u_n)_n$ is a nonnegative solution to (P_λ) for $\lambda \in (0, \lambda^*)$.*

Proof.
In what follows, we will show:

$$I'_\lambda(u_\lambda) = 0 \quad \text{and} \quad u_\lambda \neq 0, \qquad \forall \lambda \in (0, \lambda^*),$$

which imply that Lemma 3 holds true.
Firstly note that

$$I_\lambda(u_n) - \frac{1}{\gamma} I'_\lambda(u_n) u_n = I_\infty + o_n(1),$$

thus, since $uf(u) \geq 0$, for every $u \geq 0$, we get

$$\sum_{i=1}^{N} \left(\frac{1}{p_i} - \frac{1}{\gamma} \right) \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i}^{p_i} dx - \frac{C_2 \lambda}{q} \|u_n\|_q^q + \frac{1}{\gamma} \int_\Omega \left(u_n g(u_n) - \gamma G(u_n) \right) \, dx \leq I_\infty + o_n(1).$$

Since $\gamma > p^+$ and applying (g_2), it follows that

$$-\frac{C_2 \lambda}{q} \|u_n\|_q^q \leq I_\infty + o_n(1). \tag{2.7}$$

Combining the compact embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$ and the inequality (2.7), we obtain

$$-\frac{C_2 \lambda}{q} \|u_\lambda\|_q^q \leq I_\infty < 0$$

and consequently $u_\lambda \neq 0$.
To conclude that u is a solution to (P_λ), we use a result due to El Hamidi & Rakotoson [1] which implies

$$\nabla u_n(x) \longrightarrow \nabla u_\lambda(x) \quad \text{a.e. in} \quad \Omega.$$

Using the above limit, it is easy to show that u_λ is a nontrivial solution to (P_λ). ■

Proof of Theorem 1. The theorem follows directly from Lemmas 1, 2 and 3. ■

3 Proof of Theorem 2

Recall that in this section, we study the special case

$$g(u) = |u|^{p^*-2}u.$$

We will use the Mountain Pass Theorem due to Ambrosetti & Rabinowitz without the Palais Smale condition [13]. The proof of Theorem 2 will be also divided into several lemmas.

Lemma 4. *The function I_λ satisfies the geometric conditions of the Mountain Pass Theorem for all $\lambda > 0$, that is,*
a) *There exist $r, \rho > 0$ such that*

$$I_\lambda(u) \geq r \quad \textit{for} \quad \|u\| = \rho$$

b) *There exists $e_\lambda \in W_0^{1,\overrightarrow{p}}(\Omega)$ with $\|e_\lambda\| \geq \rho$ such that*

$$I_\lambda(e_\lambda) < 0.$$

Proof. Using (f_2), (f_3) and repeating the same type of arguments explored in the proof of Lemma 1, we can conclude that $a)$ holds true.
To prove $b)$, fix a nonnegative function $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \setminus \{0\}$ and note that

$$I_\lambda(t\varphi) \leq \sum_{i=1}^{N} \int_\Omega \frac{t^{p_i}}{p_i} \left| \frac{\partial \varphi}{\partial x_i} \right|^{p_i} dx - \frac{t^{p^*}}{p^*} \int_\Omega \varphi^{p^*} dx.$$

For t sufficiently large, we have

$$I_\lambda(t\varphi) < 0.$$

From the above considerations, there exists $t_\lambda > 0$ such that

$$I_\lambda(t_\lambda \varphi) = \max_{t \geq 0} I_\lambda(t\varphi).$$

Considering $v_\lambda = t_\lambda \varphi$, we have the following equality

$$\textbf{i}) \ I_\lambda(v_\lambda) = \max_{t\geq 0} I_\lambda(tv_\lambda) \quad \text{and} \quad \textbf{ii}) \ I_\lambda(v_\lambda) \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty. \tag{3.8}$$

Proof of (3.8):

The function $I_\lambda(tv_\lambda)$ has a maximum value at some point $t_0 > 0$. Since $v_\lambda \neq 0$, we have

$$\frac{\lambda \int_\Omega f(t_0 v_\lambda) t_0 v_\lambda dx + t_0^{p^*} \|v_\lambda\|_{p^*}^{p^*}}{\sum_{i=1}^{N} a_i t_0^{p_i}} = 1,$$

where

$$a_i = \left\| \frac{\partial v_\lambda}{\partial x_i} \right\|_{p_i}^{p_i}.$$

On the other hand, from definition of v_λ

$$\frac{\lambda \int_\Omega f(v_\lambda) v_\lambda \, dx + \|v_\lambda\|_{p^*}^{p_*}}{\sum_{i=1}^{N} a_i} = 1,$$

thus from $(f_4 - a)$, it follows that $t_0 = 1$, then (3.8-i) is showed. To prove (3.8-ii), note that

$$\frac{\int_\Omega f(t_\lambda \varphi) \, dx}{\sum_{i=1}^{N} b_i t_\lambda^{p_i}} \leq \frac{1}{\lambda},$$

where

$$b_i = \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{p_i}^{p_i}.$$

Thus,

$$\frac{\int_\Omega f(t_\lambda \varphi) \, dx}{\sum_{i=1}^{N} b_i t_\lambda^{p_i}} \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty$$

and by $(f_4 - b)$,

$$t_\lambda \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty.$$

Recalling that

$$0 \leq I_\lambda(v_\lambda) \leq \sum_{i=1}^{N} \int_\Omega \frac{t_\lambda^{p_i}}{p_i} \Big| \frac{\partial \varphi}{\partial x_i} \Big|^{p_i} dx - \frac{t_\lambda^{p^*}}{p^*} \int_\Omega \varphi^{p^*} dx$$

it follows that

$$I_\lambda(v_\lambda) \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty.$$

Hereafter, we fix $t_\lambda^* > 0$ sufficiently large and $e_\lambda = t_\lambda^* v_\lambda$ such that

$$\|e_\lambda\| \geq r \quad \text{and} \quad I_\lambda(e_\lambda) < 0.$$

■

Lemma 5. *If c_λ is the minimax value of the Mountain Pass Theorem applied to the functional I_λ, we have that*

$$c_\lambda \longrightarrow 0 \quad \textit{as} \quad \lambda \longrightarrow +\infty.$$

Proof.

The minimax value c_λ is given by

$$c_\lambda = \inf_{\alpha \in \Gamma} \max_{t \in [0,1]} I_\lambda(\alpha(t)),$$

where

$$\Gamma = \Big\{ \alpha \in C(W_0^{1,\overrightarrow{p}}(\Omega), \mathbb{R}); \;\; \alpha(0) = 0 \quad \text{and} \quad \alpha(1) = e_\lambda \Big\}.$$

Considering $\alpha(t) = te_\lambda$, we have that $\alpha \in \Gamma$ and

$$\max_{t \in [0,1]} I_\lambda(\alpha(t)) = \max_{t \geq 0} I_\lambda(tv_\lambda) = I_\lambda(v_\lambda)$$

then,

$$0 \leq c_\lambda \leq I_\lambda(v_\lambda),$$

from where follows

$$c_\lambda \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty.$$

■

Hereafter, we will denote by $S > 0$ the following positive constant

$$S = \inf_{u \in \mathcal{D}^{1,\overrightarrow{p}}(\mathbb{R}^N), \|u\|_{p^*}=1} \left\{ \sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} \right\}. \quad \text{(see} \quad [3])$$

The next result is an immediate consequence of the last lemma.

Lemma 6. *There exists $\lambda^* > 0$ such that*

$$0 < c_\lambda < \left(\frac{1}{p_+} - \frac{1}{p^*} \right) \min \left\{ S^{\frac{p^*}{p^* - p_+}}, S^{\frac{p^*}{p^* - p_-}} \right\}, \quad \forall\, \lambda \geq \lambda^*.$$

Related to the Mountain Pass level c_λ, there exists $(u_n)_n \subset W_0^{1,\overrightarrow{p}}(\Omega)$ satisfying

$$I_\lambda(u_n) \longrightarrow c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \longrightarrow 0.$$

Using (f_5) and standard arguments, we have that $(u_n)_n$ is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$, hence we can assume that there exists $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\overrightarrow{p}}(\Omega).$$

Lemma 7 *The weak limit u is a nontrivial solution of (P_λ).*

Proof. In the next, we will show that

$$u \neq 0 \qquad \text{and} \qquad I'_\lambda(u) = 0.$$

The equality $I'_\lambda(u) = 0$ follows of the fact that, for some subsequence, still denoted by $(u_n)_n$, we have

$$\nabla u_n(x) \longrightarrow \nabla u(x) \quad \text{a.e. in} \quad \Omega. \quad (\textit{see} \quad [3])$$

Thus, we will prove only that $u \neq 0$. Assume by contradiction that $u = 0$ and that for some subsequence

$$\|u_n\|_{p^*}^{p^*} \longrightarrow L. \tag{3.9}$$

Using the fact that $I'_\lambda(u_n)u_n \longrightarrow 0$, it follows

$$\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i}^{p_i} \longrightarrow L. \tag{3.10}$$

Since $I_\lambda(u_n) \longrightarrow c_\lambda$, it follows from (3.9)-(3.10):

$$c_\lambda \geq \left(\frac{1}{p_+} - \frac{1}{p^*} \right) L. \tag{3.11}$$

On the other hand, since $p_i > 1$ for $i = 1, \cdots, N$, by [3],

$$S\|u_n\|_{p^*}^{p_+} \leq \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} \quad \text{if} \quad \|u_n\|_{p^*} \leq 1 \tag{3.12}$$

or

$$S\|u_n\|_{p^*}^{p_-} \leq \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} \quad \text{if} \quad \|u_n\|_{p^*} \geq 1. \tag{3.13}$$

From (3.9)-(3.10) and (3.12)-(3.13), if $L > 0$ it follows that

$$L \geq \min \left\{ S^{\frac{p^*}{p^*-p_+}}, S^{\frac{p^*}{p^*-p_-}} \right\}, \tag{3.14}$$

thus from (3.11) and (3.14),

$$c_\lambda \geq \left(\frac{1}{p_+} - \frac{1}{p^*} \right) \min \left\{ S^{\frac{p^*}{p^*-p_+}}, S^{\frac{p^*}{p^*-p_-}} \right\}$$

obtaining this way a contradiction with Lemma 6. Thus, $L = 0$ and

$$u_n \longrightarrow 0 \quad \text{in} \quad W_0^{1,\overrightarrow{p}}(\Omega).$$

The last limit implies that $c_\lambda = 0$ which is an absurd, because $c_\lambda > 0$. Consequently $u \neq 0$, and the proof is finished. ■

Proof of Theorem 2. The theorem follows directly from Lemmas 4, 5, 6 and 7. ■

Remark 2. The arguments explored in this work can be used to prove that problem

$$(P) \qquad \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = f(u), & \Omega \\ u \geq 0, & \Omega \\ u = 0, & \partial\Omega \end{cases}$$

has a nonnegative solution, assuming that $(f_2) - (f_5)$. Note that, this class of problem includes the class considered by Fragala, Gazzola and Kawohl [4].

4 Regularity of weak Solutions

In this section, we show that every weak solution $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ of the following problem

$$(Q_\lambda) \qquad \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = h(x,u), & \Omega \\ u \geq 0, & \Omega \\ u = 0, & \partial\Omega \end{cases}$$

is a *strong solution*, that is, $u \in L^\infty(\Omega)$, under some hypothesis on the function h.

Lemma 4.1 *Suppose h satisfies the following growth condition:*
(h_1) There exist $c_1, c_2 \geq 0$ and $q \in (1, p^*)$ such that

$$|h(x,u)| \leq c_1 u^{q-1} + c_2 u^{p^*-1}, \quad \forall\, u \geq 0.$$

Then every weak solution $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ of (Q_λ) belongs to $L^r(\Omega)$ for all $1 \leq r < +\infty$.

Notice that under the hypothesis (h_1), Problem (Q_λ) includes the problem (P_λ).

Proof. In the next, we will use similar arguments developed by Fragala, Gazzola & Kawohl [4].

Let u be a weak solution to (Q_λ). The assertion that u belongs to $L^r(\Omega)$ for all $1 \leq r < +\infty$ may be equivalently reformulated as

$$u \in L^{(a+1)p^*}(\Omega) \quad \text{for all} \quad a > 0. \tag{4.15}$$

To prove (4.15) it is enough to show that $u^{a+1} \in W_0^{1,\overrightarrow{p}}(\Omega)$, which is in turn equivalent to

$$\lim_{L \to +\infty} \sum_{i=1}^{N} \Big(\int_\Omega |\partial_i (u \min[u^a, L])|^{p_i} \Big)^{\frac{1}{p_i}} < +\infty. \tag{4.16}$$

For each L there exists an index j such that

$$\sum_{i=1}^{N} \Big(\int_\Omega |\partial_i (u \min[u^a, L])|^{p_i} \Big)^{\frac{1}{p_i}} \leq C \Big(\int_\Omega |\partial_j (u \min[u^a, L])|^{p_j} \Big)^{\frac{1}{p_j}} \tag{4.17}$$

where C is a positive constant independent of L.

Fix such an index j, and, for every $L > 0$, set $\phi_L = u \min[u^{ap_j}, L^{p_j}] \in W_0^{1,\overrightarrow{p}}(\Omega)$. Note that for every $1 \leq i \leq N$ and for almost every $x \in \Omega$,

$$|\partial_i u|^{p_i - 2} \partial_i u \partial_i \phi_L \geq \min[u^{ap_j}, L^{p_j}] |\partial_i u|^{p_i}$$

and

$$|\partial_i (u \min[u^a, L])|^{p_i} \leq (a+1)^{p_i} \min[u^{ap_i}, L^{p_i}] |\partial_i u|^{p_i}. \tag{4.18}$$

Since u is a weak solution of (P_λ), we have

$$\sum_{i=1}^{N}\int_\Omega \min[u^{ap_j}, L^{p_j}]|\partial_i u|^{p_i} \le \sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \phi_L = \int_\Omega h(x,u)\phi_L$$

Using (h_1), for $L \ge k^a \ge 1$, it follows that

$$|h(x,u)|\phi_L \le \tilde{C}_k, \quad \text{if} \quad u < k$$

hence

$$\int_{\Omega\cap[u<k]} h(x,u)\phi_L \le \tilde{C}_k|\Omega| = C_k. \tag{4.19}$$

On the other hand,

$$|h(x,u)|\phi_L \le Cu^{p^*}\min[u^{ap_j}, L^{p_j}] \quad \text{if} \quad u \ge k$$

thus,

$$\int_{\Omega\cap[u\ge k]} h(x,u)\phi_L \le C\int_{\Omega\cap[u\ge k]} u^{p^*}\min[u^{ap_j}, L^{p_j}]. \tag{4.20}$$

From (4.19)-(4.20),

$$\int_\Omega h(x,u)\phi_L \le C_k + C\int_{\Omega\cap[u\ge k]} u^{p^*}\min[u^{ap_j}, L^{p_j}],$$

and so,

$$\sum_{i=1}^{N}\int_\Omega \min[u^{ap_j}, L^{p_j}]|\partial_i u|^{p_i} \le C_k + C\int_{\Omega\cap[u\ge k]} u^{p^*}\min[u^{ap_j}, L^{p_j}].$$

Repeating the same arguments explored in Fragala, Gazzola & Kawohl [4], there exists $\epsilon_k \to 0$, such that

$$\sum_{i=1}^{N}\int_\Omega \min[u^{ap_j}, L^{p_j}]|\partial_i u|^{p_i} \le C_k + \epsilon_k\Big[\sum_{i=1}^{N}\Big(\int_\Omega |\partial_i(u\min[u^a, L])|^{p_i}\Big)^{\frac{1}{p_i}}\Big]^{p_i}$$

which implies

$$\int_\Omega \min[u^{ap_j}, L^{p_j}]|\partial_i u|^{p_j} \le C_k + \epsilon_k\Big[\sum_{i=1}^{N}\Big(\int_\Omega |\partial_i(u\min[u^a, L])|^{p_i}\Big)^{\frac{1}{p_i}}\Big]^{p_i}. \tag{4.21}$$

From (4.18) and (4.21),

$$\int_\Omega |\partial_j(u\min[u^a,L])|^{p_j} \le C_k + \epsilon_k\Big[\sum_{i=1}^N \Big(\int_\Omega |\partial_i(u\min[u^a,L])|^{p_i}\Big)^{\frac{1}{p_i}}\Big]^{p_i}$$

and by (4.17),

$$\int_\Omega |\partial_j(u\min[u^a,L])|^{p_j} \le C_k + \epsilon_k\int_\Omega |\partial_j(u\min[u^a,L])|^{p_j}.$$

Choosing k sufficiently large, that is ε_k sufficiently small, the last inequality ensures that the integral $\int_\Omega |\partial_j(u\min[u^a,L])|^{p_j}$ is bounded for L large enough, from where follows (4.16). Hence, we conclude that every weak solution $u\in W_0^{1,\overrightarrow{p}}(\Omega)$ of (Q_λ) belongs to $L^r(\Omega)$ for all $r\ge 1$. ∎

Proposition 4.1 *Under the assumption (h_1), every solution $u\in W_0^{1,\overrightarrow{p}}(\Omega)$ of (Q_λ) belongs to $L^\infty(\Omega)$.*

Proof. For $u\ge 0$ solution of (Q_λ), we set $A_\tau=\{x\in\Omega,\ u(x)\ge\tau\}$ and $|A_\tau|$ its Lebesgue measure. Recall that the Cavalieri principle, based on the Fubini theorem, gives:

$$\int_k^{+\infty}|A_\tau|\,d\tau=\int_\Omega (u-k)_+\,dx \quad \forall\, k\ge 0.$$

Since $p^*>p^+$, one can choose $\ell>p^*$ so that

$$\beta := -\frac{1}{p^*}+\left(1-\frac{p^*}{\ell}\right)\left(1-\frac{1}{p^*}\right)\frac{1}{p^+-1}>0,$$

Let $\varphi_k=(u-k)_+$, for $k>0$ fixed. Choosing this function as a test function, combining the Cavalieri principle and the Hölder inequality, one gets

$$\begin{aligned}\sum_{i=1}^N\left\|\frac{\partial\varphi_k}{\partial x_i}\right\|_{p_i}^{p_i} &= \int_\Omega h(x,u)\varphi_k\,dx\\ &\le c_1\int_\Omega u^{q-1}\varphi_k\,dx+c_2\int_\Omega u^{p^*-1}\varphi_k\,dx\\ &\le c\left(\int_{[u\le 1]}\varphi_k\,dx+\int_{[u\ge 1]}|u|^{p^*-1}\varphi_k\,dx\right)\\ &\le c\left(|A_k|^{1-\frac{1}{p^*}}+|A_k|^{\left(1-\frac{p^*}{\ell}\right)\left(1-\frac{1}{p^*}\right)}\right)\|\varphi_k\|_{p^*} \qquad (4.22)\end{aligned}$$

Since $\lim_{k\to+\infty} \|\varphi_k\|_{p^*} = 0$, then for $k \geq k_0 > 0$, $\|\varphi_k\|_{p^*} \leq 1$.
Thus relations (3.12) and (4.22) imply

$$S\|\varphi_k\|_{p^*}^{p^+} \leq \sum_{i=1}^{N} \left\| \frac{\partial \varphi_k}{\partial x_i} \right\|_{p_i}^{p_i} \tag{4.23}$$

$$\leq c\left(|A_k|^{1-\frac{1}{p^*}} + |A_k|^{\left(1-\frac{p^*}{\ell}\right)\left(1-\frac{1}{p^*}\right)} \right) \|\varphi_k\|_{p^*} \tag{4.24}$$

Thus, for $k \geq k_0$:

$$\|\varphi_k\|_{p^*} \leq c\left(|A_k|^{1-\frac{1}{p^*}} + |A_k|^{\left(1-\frac{p^*}{\ell}\right)\left(1-\frac{1}{p^*}\right)} \right)^{\frac{1}{p^+-1}}$$

$$\leq c\left(|A_k|^{\left(1-\frac{1}{p^*}\right)\left(\frac{1}{p^+-1}\right)} + |A_k|^{\left(1-\frac{p^*}{\ell}\right)\left(1-\frac{1}{p^*}\right)\left(\frac{1}{p^+-1}\right)} \right) \tag{4.25}$$

By Cavalieri's principle, Hölder's inequality and relation(4.25), one has, for all $k \geq k_0$:

$$\int_k^{+\infty} |A_\tau|\, d\tau = \int_\Omega (u-k)_+\, dx \tag{4.26}$$

$$\leq |A_k|^{1-\frac{1}{p^*}} \|\varphi_k\|_{p^*} \tag{4.27}$$

$$\leq c\left(|A_k|^{1+\frac{1}{p^*}\frac{p^*-1}{p^+-1}} + |A_k|^{1+\beta} \right). \tag{4.28}$$

Since

$$\gamma := \frac{1}{p^*}\frac{p^*-1}{p^+-1} \geq \beta,$$

then,

$$\int_k^{+\infty} |A_\tau|\, d\tau \leq c + |A_k|^{1+\gamma}.$$

This last relation is a Gronwall type inequality, which shows that there is $d_\lambda > 0$ such that:

$$\|u\|_\infty \leq d_\lambda.$$

■

5 On sub and super-solutions

In this section, we will adapt some classical tools concerning sub and super-solutions for a class of problems involving the anisotropic operator considered above. In the case of the standard laplacian operator, we can refer to [9].

Consider the anisotropic problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = h(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5.29}$$

where Ω is a bounded domain with smooth boundary, and $h : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory satisfying the property that for each $A > 0$ fixed, there exists $C_1 > 0$ such that

$$(h_2) \qquad |h(x,t)| \leq C_1 \quad \forall (x,t) \in \Omega \times [-A, A]$$

We recall here that $W^{1,\vec{p}}(\Omega)$ is not than the completion of $\mathcal{D}(\overline{\Omega})$ with respect to the norm

$$||u|| := \|u\|_{1,\vec{p}} + ||u||_{p^+}.$$

Definition 5.1 *A function $u \in W^{1,\vec{p}}(\Omega)$ is a (weak) sub-solution to (5.29) if $u \leq 0$ on $\partial\Omega$ and for all $\varphi \in \mathcal{D}(\Omega)$ with $\varphi(x) \geq 0 \quad \forall x \in \Omega$, we have that*

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\, dx - \int_\Omega h(x,u)\varphi\, dx \leq 0.$$

Similarly, $u \in W^{1,\vec{p}}(\Omega)$ is a (weak) super-solution to (5.29) if in the above the reverse inequalities hold.

Theorem 5.1 *Suppose (h_2) holds, $\underline{u} \in W^{1,\vec{p}}(\Omega)$ is a sub-solution while $\overline{u} \in W^{1,\vec{p}}(\Omega)$ is a super-solution to problem (5.29) and assume that with constants $\underline{c}$, $\overline{c} \in \mathbb{R}$ there holds $\underline{c} \leq \underline{u} \leq \overline{u} \leq \overline{c}$ almost everywhere in Ω. Then, there exists a weak solution $u \in W_0^{1,\vec{p}}(\Omega)$ of (5.29), satisfying the condition $\underline{u} \leq u \leq \overline{u}$ almost everywhere in Ω.*

Proof. Hereafter, $H(x,u) = \int_0^u h(x,s)\,ds$ denotes the primitive of h and $J : W_0^{1,\vec{p}}(\Omega) \to I\!\!R$ the Euler-Lagrange functional associated to (5.29) given by

$$J(u) := \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \int_\Omega H(x,u)\,dx.$$

We introduce the closed and convex subset $\mathcal{M}$ of $W_0^{1,\vec{p}}(\Omega)$ defined by

$$\mathcal{M} = \Big\{ u \in W_0^{1,\vec{p}}(\Omega) \quad : \quad \underline{u} \leq u \leq \overline{u} \ \text{ a.e. in } \ \Omega \Big\}.$$

Since $\underline{u}$, $\overline{u} \ \in L^\infty(\Omega)$ by assumption, also $\mathcal{M} \subset L^\infty(\Omega)$ and consequently there exists $c > 0$ such that $|H(x,u(x))| \leq c$ for all $u \in \mathcal{M}$ and almost all $x \in \Omega$. Then,

$$J(u) \geq \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - c \ \mathrm{meas}(\Omega)$$

on $\mathcal{M}$, which implies that J is coercive on $\mathcal{M}$. We claim that the functional J is weakly lower semi-continuous on $\mathcal{M}$. Indeed, let $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$, where u_n, $u \in \mathcal{M}$. Up to a subsequence, we may consider that $u_n \longrightarrow u$ pointwise almost everywhere; moreover, $|H(x,u_n(x))| \leq c$ uniformly. Hence, the dominated convergence theorem of Lebesgue implies that

$$\int_\Omega H(x,u_n)\,dx \quad \longrightarrow \quad \int_\Omega H(x,u)\,dx, \quad \text{as } n \text{ tends to } +\infty,$$

this ends the claim, since the functional

$$u \in W_0^{1,\vec{p}}(\Omega) \longmapsto \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx$$

is clearly weakly lower semi-continuous on the all space. Once that, $W_0^{1,\vec{p}}(\Omega)$ is a reflexive Banach space, there is $u \in \mathcal{M}$ such that $J(u) = \inf_{v \in \mathcal{M}} J(v)$. We claim that u is a solves weakly (5.29), that is $J'(u) = 0$. Indeed, for every $\varphi \in \mathcal{D}(\Omega)$ and $\varepsilon > 0$ let the function $v_\varepsilon \in \mathcal{M}$ defined on Ω by:

$$v_\varepsilon(x) = \begin{cases} \overline{u}(x) & \text{if} \ \ u(x) + \varepsilon\varphi(x) \geq \overline{u}(x), \\ u(x) + \varepsilon\varphi(x) & \text{if} \ \ \underline{u}(x) \leq u(x) + \varepsilon\varphi(x) \leq \overline{u}(x), \\ \underline{u}(x) & \text{if} \ \ u(x) + \varepsilon\varphi(x) \leq \underline{u}(x). \end{cases}$$

The function v_ε can be characterized by $v_\varepsilon = (u+\varepsilon\varphi) - (\overline{\varphi}_\varepsilon - \underline{\varphi}_\varepsilon)$, where $\overline{\varphi}_\varepsilon = \max\{0, u+\varepsilon\varphi-\overline{u}\} \geq 0$ et $\underline{\varphi}_\varepsilon = -\min\{0, u+\varepsilon\varphi-\underline{u}\} \geq 0$. Note that $\underline{\varphi}_\varepsilon$, $\overline{\varphi}_\varepsilon \in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$. Since u minimizes J on $\mathcal{M}$ and J is differentiable, then

$$0 \leq J'(u)(v_\varepsilon - u) = \varepsilon J'(u)(\varphi) + J'(u)(\underline{\varphi}_\varepsilon) - J'(u)(\overline{\varphi}_\varepsilon),$$

so that

$$J'(u)(\varphi) \geq \frac{1}{\varepsilon}\left(J'(u)(\overline{\varphi}_\varepsilon) - J'(u)(\underline{\varphi}_\varepsilon)\right). \tag{5.30}$$

Now, since $\overline{u}$ is a supersolution to (5.29), we get

$$\begin{aligned}
J'(u)(\overline{\varphi}_\varepsilon) &= J'(\overline{u})(\overline{\varphi}_\varepsilon) + [J'(u) - J'(\overline{u})](\overline{\varphi}_\varepsilon)\\
&\geq [J'(u) - J'(\overline{u})](\overline{\varphi}_\varepsilon)\\
&= \sum_{i=1}^N \int_{\Omega_\varepsilon}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\right)\frac{\partial}{\partial x_i}(u-\overline{u}+\varepsilon\varphi)\,dx -\\
&\quad -\int_{\Omega_\varepsilon}[h(x,u)-h(x,\overline{u})](u-\overline{u}+\varepsilon\varphi)\,dx\\
&\geq \varepsilon\sum_{i=1}^N \int_{\Omega_\varepsilon}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\right)\frac{\partial}{\partial x_i}(\varphi) -\\
&\quad -\varepsilon\int_{\Omega_\varepsilon}|h(x,u)-h(x,\overline{u})|\,|\varphi|\,dx
\end{aligned}$$

where $\Omega_\varepsilon = \{x\in\Omega \,:\, u(x)+\varepsilon\varphi(x) \geq \overline{u}(x)\}$. Notice that $\text{meas}(\Omega_\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Then

$$J'(u)(\overline{\varphi}_\varepsilon) \geq \text{o}(\varepsilon),$$

where $\text{o}(\varepsilon)/\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Similarly, we conclude that

$$J'(u)(\underline{\varphi}_\varepsilon) \leq \text{o}(\varepsilon),$$

and consequently, with (5.30), we get

$$J'(u)(\varphi) \geq 0$$

for every $\varphi \in \mathcal{D}(\Omega)$. This implies, by reversing the sign of φ, that $J'(u)(\varphi) = 0$ for every $\varphi \in \mathcal{D}(\Omega)$. We conclude the proof using the density of $\mathcal{D}(\Omega)$ in $W_0^{1,\vec{p}}(\Omega)$. ■

An example of application.

Consider the special case $h(x,u) = f(x)(\cos u + 2) - \sum_{i=1}^{l} k_i(x)|u|^{\alpha_i - 2}u$ where f,k_i belong to $L^\infty(\Omega)$ for $i = \{1, ...l\}$ and satisfy

$$f(x) > 0 \quad \text{and} \quad f(x) - \sum_{i=1}^{l} k_i(x) < 0 \quad \forall x \in \Omega.$$

Note that the functions $\underline{u} = 0$ and $\overline{u} = 1$ are sub-solution and super-solution for this case. Using Theorem 5.1 and the above function h, it follows that (5.29) has a solution $u \in W_0^{1,\vec{p}}(\Omega)$.

References

[1] A. El Hamidi, J. M. Rakotoson, Compactness and quasilinear problems with critical exponents, *Diff. Int. Equ.* 18 (2005) 1201–1220.

[2] A. El Hamidi, J. M. Rakotoson, On a pertubed anisotropic equation with a critical exponent, *Richerche di Matematica* 55 (2006) 55–69.

[3] A. El Hamidi, J. M. Rakotoson, Extremal functions for the anisotropic Sobolev inequalities, *Ann. I. H. Poincaré* AN 24 (2007) 741–756.

[4] I. Fragala, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear elliptic equation, *Ann. I. H. Poincaré* AN 21 (2004) 715–734.

[5] P.L. Lions, The concentration-compactness principle in the calculus of variations. The limit case, *part 1Rev. Mathemática Iberoamericana* Vol

1 No 1 (1985), 145-201. *part 2 Rev. Mathemática Iberoamericana* Vol 1 No 2(1985), 45–121.

[6] S.M. Nikol'skii, On embedding continuation and approximation theorems for differentiable functions of several variables, *Russian Math. Surv.* 16 (1961) 55-104.

[7] J. Rakosnik, Some remarks to anisotropic Sobolev spaces I, *Beiträge Anal.* 13 (1979) 55-68.

[8] J. Rakosnik, Some remarks to anisotropic Sobolev spaces II, *Beiträge Anal.* 15 (1981) 127-140.

[9] M. Struwe, Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems, *Springer-Verlag*, (1996)

[10] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* (4) 110 (1976) 353-372.

[11] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche Mat.* 18 (1969) 3-24.

[12] L. Ven'-tuan, On embedding theorems for spaces of functions with partial derivatives of various degrees of summability. *Vestnik Leningrad Univ.* 16 (1961), pp. 23-37 (in Russian) .

[13] M. Willem, Minimax Theorems. *Birkhäuser*, 1996.