Nonexistence results of solutions to systems of semilinear differential inequalities on the Heisenberg group

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Abstract

We establish nonexistence results to systems of differential inequalities on the $(2N+1)$-Heisenberg group. The systems considered here are of the type (ES_m). These nonexistence results hold for N less than critical exponents which depend on p_i and γ_i, $1 \leq i \leq m$. Our results improve the known estimates of the critical exponent.

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1 Introduction

For the reader's convenience, we recall some background facts used here. The Heisenberg group $\mathbb{H}^N$, whose points will be denoted by $\eta = (x, y, \tau)$ is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(< x\,,\, \tilde{y} > - < \tilde{x}\,,\, y >)),$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^N$. The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}^N$ is obtained, from the vector fields $X_i = \partial_{x_i} + 2y_i\,\partial_\tau$ and $Y_i = \partial_{y_i} - 2x_i\,\partial_\tau$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N}(X_i^2 + Y_i^2). \tag{1}$$

Observe that the vector field $T = \partial_\tau$ does not appear in (1). This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \;\; j,\, k \in \{1, 2, ..., N\}. \tag{2}$$

The relation (2) proves that $\mathbb{H}^N$ is a nilpotent Lie group of order 2. Incidently, (2) constitues an abstract version of the canonical relations of commutation of Heisenberg between momentums and positions. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).$$

A natural group of dilatations on $\mathbb{H}^N$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \quad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where

$$Q = 2N + 2$$

is the homogeneous dimension of $\mathbb{H}^N$.

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}^N$ and homogeneous w.r.t the dilatations δ_λ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}} u)(\eta \circ \tilde{\eta}), \ \ \forall (\eta, \tilde{\eta}) \in \mathbb{H}^N \times \mathbb{H}^N,$$

$$\Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda.$$

It is natural to define a distance from η to the origin by

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \sum_{i=1}^{N} (x_i^2 + y_i^2)^2 \right)^{1/4}.$$

In [7], Pohozaev and Véron gave another proof of the result of Birindelli, Capuzzo-Dolcetta and Cutri [1] concerning the nonexistence of weak solutions of the differential inequality

$$\Delta_{\mathbb{H}}(au) + |\eta|_{\mathbb{H}}^{\gamma} \, |v|^p \leq 0 \quad \text{in} \quad \mathbb{H}^N$$

for $\gamma > -2$, $1 < p \leq (Q + \gamma)/(Q - 2)$ and $a \in L^\infty(\mathbb{H}^N)$.
They then addressed the question of nonexistence of weak solutions of the system (ES_2):

$$-\Delta_{\mathbb{H}}(a_1 u) \geq |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1}, \quad -\Delta_{\mathbb{H}}(a_2 v) \geq |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2},$$

where a_i, $i \in \{1,2\}$ are measurable and bounded functions defined on $\mathbb{H}^N$, and $p_i \in (1,+\infty)$, $\gamma_i \in \mathbb{R}$, $i \in \{1,2\}$. They showed that this system admits no solution defined in $\mathbb{H}^N$ whenever $\gamma_i > -2$ and $1 < p_i \leq (Q+\gamma_i)/(Q-2)$, $i = 1$, 2. The estimates on p_i, $i = 1$, 2, are obtained using Young's inequality and they are not optimal. Using the Hölder inequality, we obtain better estimates on p_i, $1 \leq i \leq m$. The same strategy is suitable to study the systems (PS$_m$) and (HS$_m$):

$$(\mathrm{PS}_m)\ \partial u_i/\partial t - \Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}},\ \eta \in \mathbb{H}^N,\ 1 \leq i \leq m,\ u_{m+1} = u_1,$$
$$(\mathrm{HS}_m)\ \partial^2 u_i/\partial t^2 - \Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}},\ \eta \in \mathbb{H}^N,\ 1 \leq i \leq m,\ u_{m+1} = u_1,$$

to obtain the following results:

Theorem 1 *Assume that the initial data* $u_i^{(0)} \in L^1\left(\mathbb{R}^{2N+1}\right)$ *and* $\int u_i^{(0)}(\eta)\, d\eta \geq 0$, $1 \leq i \leq m$. *If*

$$Q \leq \max\{X_1, X_2, ..., X_m\},$$

where the vector $(X_1, X_2, ..., X_m)^T$ *is the solution of (16), then there is no nontrivial global weak solution* $(u_1, ..., u_m)$ *of the system (PS*$_m$*).*

Theorem 2 *Assume that initial data (for the first derivatives of* u_i, $1 \leq i \leq m$*)* $u_i^{(1)} \in L^1\left(\mathbb{R}^{2N+1}\right)$ *and* $\int u_i^{(1)}(\eta)\, d\eta \geq 0$, $1 \leq i \leq m$. *If*

$$Q \leq 1 + \max\{X_1, X_2, ..., X_m\},$$

where the vector $(X_1, X_2, ..., X_m)^T$ *is the solution of (16), then there is no nontrivial global weak solution* $(u_1, ..., u_m)$ *of the system (HS*$_m$*).*

In [2], the first author and Obeid presented results for systems of evolution type with higher-order time derivatives. Their results are the generalized versions of our previous results (Theorems 1 and 2) on (PS$_m$) and (HS$_m$). For interesting results on elliptic equations and systems, we refer to the recent articles Kartsatos and Kurta [3], Kurta [4, 5], and Mitidieri and Pohozaev [6] .

To render the presentation very clear, we start with the case of systems of two inequalities

2 Systems of two Inequalities

In this section, we will treat the case $m = 2$ and consider the system

$$(\mathrm{ES}_2)\quad \left\{ \begin{array}{rcl} -\Delta_{\mathbb{H}}(a_1 u) & \geq & |\eta|_{\mathbb{H}}^{\gamma_1}\, |v|^{p_1}, \\ \\ -\Delta_{\mathbb{H}}(a_2 v) & \geq & |\eta|_{\mathbb{H}}^{\gamma_2}\, |u|^{p_2}, \end{array} \right.$$

where a_i, $i \in \{1, 2\}$, be measurable and bounded functions defined on $\mathbb{H}^N$, $p_i > 1$ and γ_i, $i = 1, 2$ real numbers. We identify points in $\mathbb{H}^N$ with points in $\mathbb{R}^{2N+1}$. We also recall that the Haar measure on $\mathbb{H}^N$ is identical to the Lebesgue measure $d\eta = dx\, dy\, d\tau$ on $\mathbb{R}^{2N+1} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$. In the sequel, the integral $\int_{\mathbb{R}^{2N+1}}$ will be simply denoted by $\int$, the measure of integration however will be specified.

Definition 1 *Let a_1 and a_2 be two bounded measurable functions on $\mathbb{R}^{2N+1}$. A weak solution (u, v) of the system (ES_2) on $\mathbb{R}^{2N+1}$ is a pair of locally integrable functions (u, v) such that*

$$\left\{ \begin{array}{rcl} u & \in & L^{p_2}_{loc}\left(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_2}\, d\eta\right), \\ \\ v & \in & L^{p_1}_{loc}\left(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_1}\, d\eta\right) \end{array} \right.$$

satisfying

$$\int_{\mathbb{R}^{2N+1}} \left(a_1 u \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi\right)\, d\eta \leq 0, \tag{3}$$

and

$$\int_{\mathbb{R}^{2N+1}} \left(a_2 v \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi\right)\, d\eta \leq 0 \tag{4}$$

for any nonnegative test function $\varphi \in C^2_c(\mathbb{R}^{2N+1})$.

Theorem 3 *Assume that*

$$Q \leq Q_e^* = 2 + \frac{1}{p_1 p_2 - 1} \max\{(\gamma_1 + 2) + p_1(\gamma_2 + 2)\, ; p_2(\gamma_1 + 2) + (\gamma_2 + 2)\}.$$

Then there is no nontrivial weak solution (u, v) of the system (ES_2).

Proof. Let $\varphi_{_R} \in \mathcal{D}(\mathbb{H}^N)$ be a nonnegative function such that

$$\varphi_{_R}(\eta) = \Phi^{\lambda}\left(\frac{\tau^2 + |x|^4 + |y|^4}{R^4}\right), \tag{5}$$

where $\lambda >> 1$, $R > 0$ and $\Phi \in \mathcal{D}([0, +\infty[)$ is the "standard cut-off function"

$$\Phi(r) = \left\{\begin{array}{lll} 1 & \text{if} & 0 \leq r \leq 1, \\ 0 & \text{if} & r \geq 2, \end{array}\right. \qquad 0 \leq \Phi(r) \leq 1.$$

Note that $\text{supp}(\varphi_{_R})$ is a subset of

$$\Omega_R = \{\eta \equiv (x, y, \tau) \in \mathbb{H}^N;\;\; 0 \leq \tau^2 + |x|^4 + |y|^4 \leq 2R^4\}$$

and $\text{supp}(\Delta_{\mathbb{H}}\varphi_{_R})$ is included in

$$\mathcal{C}_R = \{\eta \equiv (x, y, \tau) \in \mathbb{H}^N;\;\; R^4 \leq \tau^2 + |x|^4 + |y|^4 \leq 2R^4\}.$$

Let

$$\rho = \frac{\tau^2 + |x|^4 + |y|^4}{R^4},$$

then

$$\begin{array}{l} \Delta_{\mathbb{H}}\varphi_{_R}(\eta) = \dfrac{4(N+4)\Phi'(\rho)}{R^4}\lambda\Phi^{\lambda-1}(\rho)(|x|^2 + |y|^2) + \\ \dfrac{16\Phi''(\rho)}{R^8}\lambda\Phi^{\lambda-1}(\rho)\left((|x|^6 + |y|^6) + \tau^2(|x|^2 + |y|^2) + 2\tau < x\,,\, y > (|x|^2 - |y|^2)\right) + \\ \dfrac{16\Phi'^2(\rho)}{R^8}\lambda(\lambda-1)\Phi^{\lambda-2}(\rho)\left((|x|^6 + |y|^6) + \dfrac{\tau^2}{4}(|x|^2 + |y|^2) + 2\tau < x\,,\, y > (|x|^2 - |y|^2)\right). \end{array}$$

It follows that there is a positive constant $C > 0$, independent of R, such that

$$|\Delta_{\mathbb{H}}\varphi_{_R}(\eta)| \leq \frac{C}{R^2}, \quad \forall \eta \in \Omega_R. \tag{6}$$

Let (u, v) be a nontrivial weak solution of (ES_2). Using (3) and (4) with $\varphi = \varphi_{_R}$ one has

$$\begin{array}{rcl} \displaystyle\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_{_R}\,d\eta & \leq & \displaystyle -\int a_1 u\Delta_{\mathbb{H}}\varphi_{_R}\,d\eta \leq ||a_1||_{L^\infty}\int |u|\,|\Delta_{\mathbb{H}}\varphi_{_R}|\,d\eta \\ & \leq & \displaystyle ||a_1||_{L^\infty}\left(\int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_{_R}\right)^{1/p_2}\left(\int |\Delta_{\mathbb{H}}\varphi_{_R}|^{p_2'}\,(\varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_2})^{1-p_2'}\right)^{1/p_2'} \end{array} \tag{7}$$

and

$$
\begin{aligned}
\int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi_{_R}\, d\eta &\leq -\int a_2 v \Delta_{\mathbb{H}} \varphi_{_R}\, d\eta \\
&\leq ||a_2||_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi_{_R} \right)^{1/p_1} \left(\int |\Delta_{\mathbb{H}} \varphi_{_R}|^{p_1'} \left(\varphi_{_R} |\eta|_{\mathbb{H}}^{\gamma_1} \right)^{1-p_1'} \right)^{1/p_1'} \qquad (8),
\end{aligned}
$$

thanks to the Hölder inequality. Setting

$$
I(R) = \int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi_{_R}\, d\eta \quad \text{and} \quad J(R) = \int |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi_{_R}\, d\eta,
$$

we have

$$
J(R) \leq C_1\, I(R)^{1/p_2}\, \mathcal{A}_{p_2,\gamma_2}(R)^{1/p_2'}, \tag{9}
$$

where

$$
\mathcal{A}_{p_2,\gamma_2}(R) = \int |\Delta_{\mathbb{H}} \varphi_{_R}|^{p_2'} \left(\varphi_{_R} |\eta|_{\mathbb{H}}^{\gamma_2} \right)^{1-p_2'}\, d\eta,
$$

and C_1 is a positive constant independent of R. Similarly, we have

$$
I(R) \leq C_2\, J(R)^{1/p_1}\, \mathcal{A}_{p_1,\gamma_1}(R)^{1/p_1'}, \tag{10}
$$

where

$$
\mathcal{A}_{p_1,\gamma_1}(R) = \int |\Delta_{\mathbb{H}} \varphi_{_R}|^{p_1'} \left(\varphi_{_R} |\eta|_{\mathbb{H}}^{\gamma_1} \right)^{1-p_1'}\, d\eta,
$$

and C_2 is a positive constant independent of R.

Note that for λ sufficiently large, the integrals $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, are convergent. Indeed, in the expression $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, we have $|\eta|_{\mathbb{H}} \geq R^4$, and the exponent of $\varphi_{_R}$ is positive for λ large enough.

In order to estimate the integrals $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, we introduce the scaled variables

$$
\tilde{\tau} = R^{-2}\, \tau, \quad \tilde{x} = R^{-1}\, x, \quad \tilde{y} = R^{-1}\, y. \tag{11}
$$

Using the fact that $\mathrm{supp}\varphi_{_R} \subset \Omega_R$, we conclude that

$$
\mathcal{A}_{p_i,\gamma_i}(R) \leq C\, R^{2N+2-2p_i'+\gamma_i(1-p_i')}, \quad i \in \{1,2\}. \tag{12}
$$

Using (10) and (12) in (9), we obtain

$$
J(R)^{1-\frac{1}{p_1 p_2}} \leq C\, \mathcal{A}_{p_1,\gamma_1}(R)^{\frac{1}{p_1' p_2}}\, \mathcal{A}_{p_2,\gamma_2}(R)^{\frac{1}{p_2'}} \leq C R^{\sigma_J},
$$

where

$$\begin{aligned}\sigma_J &= \frac{1}{p_2'}(2N+2-2p_2+\gamma_2(1-p_2')) + \frac{1}{p_1' p_2}(2N+2-2p_1+\gamma_1(1-p_1'))\\ &= Q\left(1-\frac{1}{p_1p_2}\right) - \frac{(2p_2+2+\gamma_2)p_1+\gamma_1}{p_1p_2}.\end{aligned}$$

Similarly, we have

$$I(R)^{1-\frac{1}{p_1p_2}} \leq C\,\mathcal{A}_{p_1,\gamma_1}(R)^{\frac{1}{p_1'}}\,\mathcal{A}_{p_2,\gamma_2}(R)^{\frac{1}{p_1p_2'}} \leq CR^{\sigma_I},$$

where

$$\sigma_I = Q\left(1-\frac{1}{p_1p_2}\right) - \frac{(2p_1+2+\gamma_1)p_2+\gamma_2}{p_1p_2}.$$

Now, we require $\sigma_I \leq 0$ or $\sigma_J \leq 0$ which is equivalent to

$$\begin{aligned}Q \leq Q_e^* &= \frac{1}{p_1p_2-1}\max\{p_1(2(p_2+1)+\gamma_2)+\gamma_1\,;\,p_2(2(p_1+1)+\gamma_1)+\gamma_2\}\\ &= 2+\frac{1}{p_1p_2-1}\max\{(\gamma_1+2)+p_1(\gamma_2+2)\,;p_2(\gamma_1+2)+(\gamma_2+2)\}.\end{aligned}$$

In this case, the integrals $I(R)$ and $J(R)$, increasing in R, are bounded uniformly with respect to R. Using the monotone convergence theorem, we deduce that $|\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}$ and $|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}$ are in $L^1(\mathbb{R}^{2N+1})$. Note that instead of (7) we have more precisely

$$\begin{aligned}\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\,d\eta &\leq ||a_1||_{L^\infty}\left(\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\,d\eta\right)^{1/p_2}\mathcal{A}_{p_2,\gamma_2}(R)^{1/p_2'}\\ &\leq C\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\,d\eta.\end{aligned}$$

Finally, using the dominated convergence theorem, we obtain that

$$\lim_{R\to+\infty}\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\,d\eta = 0.$$

Hence

$$\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\,d\eta = 0,$$

which implies that $v \equiv 0$ and $u \equiv 0$ via (8). This contradicts the fact that (u, v) is a nontrivial weak solution of (ES_2), which achieves the proof. □

Remark 1 *The critical exponent* Q_e^* *can be written as*

$$Q_e^* = 2 + \max\{X_1, X_2\},$$

where the vector $(X_1, X_2)^T$ *is the solution of the linear system*

$$\begin{pmatrix} -1 & p_1 \\ p_2 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \end{pmatrix}. \tag{13}$$

Comment 1 *In their paper, Pohozaev and Véron [7] showed that if*

$$1 < p_j \leq \frac{Q + \gamma_j}{Q - 2}, \quad j \in \{1, 2\}, \tag{14}$$

then the system (ES_2*) has no nontrivial weak solution. The condition (14) is equivalent to*

$$Q \leq 2 + \min\left\{\frac{\gamma_1 + 2}{p_1 - 1}; \frac{\gamma_2 + 2}{p_2 - 1}\right\}. \tag{15}$$

Theorem 1 gives a better estimate of the exponent. Indeed,

$$\frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1 p_2 - 1} - \frac{\gamma_2 + 2}{p_2 - 1} = -\frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1 p_2 - 1} + \frac{\gamma_1 + 2}{p_1 - 1},$$

which implies that

$$\max\left\{\frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1 p_2 - 1}; \frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1 p_2 - 1}\right\} \geq \min\left\{\frac{\gamma_1 + 2}{p_1 - 1}; \frac{\gamma_2 + 2}{p_2 - 1}\right\}.$$

3 Systems of m semilinear inequalities

In this section, we give generalizations of the last results to systems with m inequalities, $m \in \mathbb{N}^*$.

Let $(X_1, X_2, ..., X_m)$ be the solution of the linear system

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -p_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -p_{m-1} \\ -p_m & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{pmatrix} = \begin{pmatrix} -\gamma_1 - 2 \\ -\gamma_2 - 2 \\ \vdots \\ -\gamma_{m-1} - 2 \\ -\gamma_m - 2 \end{pmatrix}, \tag{16}$$

where $p_i > 1$ and γ_i are given real numbers, $i \in \{1, 2, ..., m\}$.

Consider the system (ES$_m$):

$$-\Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad \eta \in \mathbb{H}^N, \quad 1 \leq i \leq m, \quad u_{m+1} = u_1, \tag{17}$$

where $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$.

Definition 2 *Let a_i, $i \in \{1, 2, ..., m\}$, be m bounded measurable functions on $\mathbb{R}^{2N+1}$. A weak solution $(u_1, ..., u_m)$ of the system (ES$_m$) on $\mathbb{R}^{2N+1}$ is a vector of locally integrable functions $(u_1, ..., u_m)$ such that*

$$u_i \in L^{p_i}_{loc}\left(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_i}\, d\eta\right), \quad i \in \{1, 2, ..., m\},$$

satisfying

$$\int_{\mathbb{R}^{2N+1}} \left(a_i u \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}} \varphi\right) d\eta \leq 0, \quad i \in \{1, 2, ..., m-1\} \tag{18}$$

and

$$\int_{\mathbb{R}^{2N+1}} \left(a_m u_m \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_1} |u|^{p_1} \varphi\right) d\eta \leq 0 \tag{19}$$

for any nonnegative test function $\varphi \in C^2_c(\mathbb{R}^{2N+1})$.

Theorem 4 *If $Q \leq 2 + \max\{X_1, X_2, ..., X_m\}$ then the system (ES$_m$) has no nontrivial solution.*

Proof. In order to simplify the proof, we treat only the case $m = 3$; the general case can be established in the same manner.

Let (u_1, u_2, u_3) be a nontrivial weak solution of (ES$_m$). The inequalities (18) and (19), with $\varphi = \varphi_R$ defined by (5), imply that

$$\int |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi_R \, d\eta \leq ||a_3||_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} \varphi_R\right)^{1/p_3} \left(\int |\Delta_{\mathbb{H}}\varphi_R|^{p_3'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_3}\right)^{1-p_3'}\right)^{1/p_3'},$$

$$\int |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} \varphi_R \, d\eta \leq ||a_1||_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi_R\right)^{1/p_1} \left(\int |\Delta_{\mathbb{H}}\varphi_R|^{p_1'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1}\right)^{1-p_1'}\right)^{1/p_1'}$$

and

$$\int |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} \varphi_R \, d\eta \leq ||a_2||_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} \varphi_R\right)^{1/p_2} \left(\int |\Delta_{\mathbb{H}}\varphi_R|^{p_2'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2}\right)^{1-p_2'}\right)^{1/p_2'}.$$

Let

$$\begin{aligned} I_i(R) &= \int |\eta|_{\mathbb{H}}^{\gamma_i} |u_i|^{p_i} \varphi_{_R}\, d\eta, \quad 1 \leq i \leq 3, \\ \mathcal{A}_i(R) &= \int |\Delta_{\mathbb{H}} \varphi_{_R}|^{p_i'} \left(\varphi_{_R} |\eta|_{\mathbb{H}}^{\gamma_i}\right)^{1-p_i'}, \quad 1 \leq i \leq 3, \end{aligned}$$

then there is a positive constant C such that

$$\left\{ \begin{aligned} I_1 &\leq C\, I_3^{1/p_3}\, \mathcal{A}_3^{1/p_3{}'}, \\ I_2 &\leq C\, I_1^{1/p_1}\, \mathcal{A}_1^{1/p_1{}'}, \\ I_3 &\leq C\, I_2^{1/p_2}\, \mathcal{A}_2^{1/p_2{}'}. \end{aligned} \right.$$

Hence, the estimates

$$\left\{ \begin{aligned} I_1^{1-\frac{1}{p_1p_2p_3}} &\leq C \mathcal{A}_1^{\frac{1}{p_1'p_2p_3}}\, \mathcal{A}_2^{\frac{1}{p_2'p_3}}\, \mathcal{A}_3^{\frac{1}{p_3'}}, \\ I_2^{1-\frac{1}{p_1p_2p_3}} &\leq C \mathcal{A}_1^{\frac{1}{p_1'}}\, \mathcal{A}_2^{\frac{1}{p_1p_2'p_3}}\, \mathcal{A}_3^{\frac{1}{p_1p_3'}}, \\ I_3^{1-\frac{1}{p_1p_2p_3}} &\leq C \mathcal{A}_1^{\frac{1}{p_1'p_2}}\, \mathcal{A}_2^{\frac{1}{p_2'}}\, \mathcal{A}_3^{\frac{1}{p_1p_2p_3'}}, \end{aligned} \right.$$

hold true.

In order to estimate the expressions I_i, $1 \leq i \leq 3$, we use the scaled variables (11) and obtain

$$I_i^{1-\frac{1}{p_1p_2p_3}} \leq C R^{\sigma_i}, \quad 1 \leq i \leq 3,$$

where

$$\left\{ \begin{aligned} \sigma_1 &= \left(1 - \tfrac{1}{p_1p_2p_3}\right) \left(Q - 2 - \tfrac{(\gamma_1+2)+p_1(\gamma_2+2)+p_1p_2(\gamma_3+2)}{p_1p_2p_3-1}\right), \\ \sigma_2 &= \left(1 - \tfrac{1}{p_1p_2p_3}\right) \left(Q - 2 - \tfrac{p_2p_3(\gamma_1+2)+(\gamma_2+2)+p_2(\gamma_3+2)}{p_1p_2p_3-1}\right), \\ \sigma_3 &= \left(1 - \tfrac{1}{p_1p_2p_3}\right) \left(Q - 2 - \tfrac{p_3(\gamma_1+2)+p_1p_3(\gamma_2+2)+(\gamma_3+2)}{p_1p_2p_3-1}\right). \end{aligned} \right.$$

Now, we require that, at least, one of σ_i, $1 \leq i \leq 3$, is less than zero, which is equivalent to $Q \leq 2 + \max\{X_1, X_2, X_3\}$, where the vector $(X_1, X_2, X_3)^T$ is the solution of

$$\begin{pmatrix} 1 & -p_1 & 0 \\ 0 & 1 & -p_2 \\ -p_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -\gamma_1 - 2 \\ -\gamma_2 - 2 \\ -\gamma_3 - 2 \end{pmatrix}.$$

Following the arguments used in the proof of Theorem 1, we conclude that $(u_1, u_2, u_3) \equiv (0, 0, 0)$. This ends the proof by contradiction. □

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